14.6.14.2 Pricing a Vanilla Cap

The market convention pricing of caps, floors, and collars has many features in common with swaptions. The key steps are:

- Create a discounting (e.g. *df* or zero-coupon) curve from traded instruments, to be used for all the PV, forward rates, and related calculations.

- Obtain the (yield) volatility for the cap. Cap vols have a duality rather like YTM's vs. zero-coupon rates. One may use a strip of vols for the individual options in the strip, called "spot vols". Alternatively, one may use a single cap vol, which is a "summary measure" vol that applies to the entire cap called "flat vol". In practice, the convention is to quote caps on flat vol, rather than on a term-structure of caplet spot vols. One must be able to convert between the two for the same reasons one must convert between zero-coupon rates and YTM's. This is discussed below, in Chapter 12, and in [8.b] and [8.c].

- Price the strip of caplets using Black76, which produces a cap premium in terms of yield, and then convert that yield based option premium to a price based option premium.

The market convention caplet and floorlet pricing formulas are the usual Black76 purely yield-based forms as:

$$CapletY_{Black76,Y}\Big|_i = e^{-r_i t_i}\left[F_{Y,i}N\left(d_{1,i}\right) - K_{Y,i}N\left(d_{2,i}\right)\right] \tag{14.38}$$

and

$$FloorletY_{Black76,Y}\Big|_i = -e^{-r_i t_i}\left[F_{Y,i}N\left(-d_{1,i}\right) - K_{Y,i}N\left(-d_{2,i}\right)\right] \tag{14.39}$$

where suffix the *YTM* and subscript *Y* denote these items are units of "yield" or "borrowing rate" such as LIBOR, the subscripts "*i*" denote that there is one of these for each caplet or floorlet in the strip, and each one has its own forward, possibly its own strike, its own PV'ing rate, and possibly its own vol if strip of vols is use. As expected:

$$d_{1,i} = \frac{\ln\left(\frac{F_{Y,i}}{K_{Y,i}}\right) + \left({\sigma_{Y,i}}^2 / 2\right) t_i}{\sigma_{Y,i}\sqrt{t_i}}, \quad and \quad d_{2,i} = d_{1,i} - \sigma_{Y,i}\sqrt{t_i}$$

The inputs to the "pure idealised" yield-based BSM model are defined as:

$F_{Y.i}$: the forward "rate" for the i^{th} forward period.

$K_{Y.i}$: the caplet/floorlet strike expressed in terms of forward rate for the i^{th} forward period. In the vanilla case, all the *K's* are the same, and equal to the cap or floor rate.

$r_{.i}$: the rate required to PV the pay-out to today for the i^{th} forward period.

$\sigma_{Y.i}$: volatility of the forward rate or yield either for the i^{th} forward period, or the (summary) cap or floor vol for the entire structure, as is commonly quoted in the markets.

$t_{.i}$: the time to expiration (unrelated to the time to Maturity, but related to the forward date) for the i^{th} forward period.

As with swaptions, Equations (14.38) and (14.39) cannot be used directly to price options since if the units of the parameters are yield-based, then the resulting option premium is also yield-based. The yield-based result is converted to a price-based/cash result with a PV'ing conversion appropriate for caps, and parallels the annutisation conversion for swaps. In particular, the caplet/floorlet premium will be in units of the "term deposit/loan" which is a money market yield, and typically quarterly or semi-annual. That is, the "yield-based premium" will (usually) be a money market rate for the period of the deposit/loan, and is converted to an annual cash equivalent using results from Chapter 12or [8.b]. For example:

$$Caplet_{Black76,P}\Big|_i = Notional_i a_i \; CapletY_{Black76,Y}\Big|_i \tag{14.40}$$

where a_i is the conversion factor, as per

$$Caplet_{Black76,P}\Big|_i = Notional_i \left(\frac{\frac{Days_i}{Basis}}{1+F_i \frac{Days_i}{Basis}} \right) CapletY_{Black76,Y}\Big|_i \tag{14.41}$$

and

$$Floorlet_{Black76,P}\Big|_i = Notional_i \left(\frac{\frac{Days_i}{Basis}}{1+F_i \frac{Days_i}{Basis}} \right) FloorletY_{Black76,Y}\Big|_i \tag{14.42}$$

where F_i is the (forward) rate for the i^{th} forward period that was also the "underlying price" for the caplet, and $Days_i$ is the number of days in the i^{th} forward period, and $Notional_i$ is the notional value of the trade for the i^{th} forward period, though for vanilla caps this is uniform for each period.

Now, price an 18-month cap against 6-month LIBOR struck at 3.90% on 25,000,000 notional, where the market information is as follows, with the dates converted to days from today (Days) and year fractions (Alphas) on an Act/360 basis, assuming that all dates "match[406]":

Index	Days	DT	Market	Alpha	AlphaF	df
6mLIBOR	183		3.23%	0.508333		0.983846067
12mLIBOR	364		3.51%	1.011111	0.502778	0.965726371
12x18FRA	545	181	3.87%	1.513889	0.502778	0.94729439

Remember, LIBOR rates are spot cash deposit rates, while FRA's are forward rates.

The LIBOR *df's* are arrived at via $df_i = \frac{1}{1+y_i * \alpha_i}$, where $\alpha_i = {Days_i}/{Basis}$. The FRA based *df* is arrived at by "bootstrapping" the FRA rate with the 12-month *df*, where FRA "Days" is the number of days to FRA maturity, and FRA DT is the number of days in the FRA period[407] (i.e. Maturity – Fixing), as:

$$df_{18} = df_{12} df_{12x18} = df_{12}\left(\frac{1}{1+y_{12x18} * \alpha_{18-12,Basis}}\right)$$
$$= 0.962905664\left(\frac{1}{1+3.87\% * 0.502778}\right) = 0.94729439$$

that is, obtain the forward *df* directly from the FRA, and then PV the forward *df* to obtain the 18-month spot *df*.

All the PV'ing and forward rate calculations will be derived from this market implied *df*-curve.

Step 1: The Forward Rates

The next step is to determine the "underlying price" for each of the caplets. These correspond to the forward rates for the relevant forward periods of each caplet. Although an 18-month cap against a 6-month index sounds like a 3-period problem, there are only two forward periods for caplets. This occurs since the first 6-month period does not include a caplet for vanilla caps at inception (since the first period's rate is fixed "today", so it is known with certainty, and a caplet is worthless).

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[406] That is, it is assumed that all the dates in all the caplets etc all correspond to dates with exactly these numbers of days between them. In reality, there may be some small differences (e.g. 1 or days) due to specific date/convention issues relating to settlement offset days, holidays, etc., see [4] or [8.b] for details. However, that minutia is omitted for clarity.

[407] See [4] or [8.b] for a detailed treatment of curve building.

The forward rate for the first caplet is implied by the first two *df's*, since in this example those are taken to be the *df's* also for the caplet dates. Thus, the first underlying caplet rate is:

$$df_{6m,6m} = \frac{df_{12m}}{df_{6m}} = \frac{1}{1 + y_{6m,6m}\alpha_{6m,6m}}$$

so

$$y_{6m,6m} = \frac{1}{\alpha_{6m,6m}}\left(\frac{df_{6m}}{df_{12m}} - 1\right) = \frac{1}{0.502778}\left(\frac{0.983846067}{0.965726371} - 1\right)$$
$$= 3.7318\%$$

This would also equate to a 6x12's FRA (if the dates matched), so if that FRA rate was available, then that could have been used directly.

The second caplet's forward rate for this example can be taken directly from the 12x18's FRA rate as 3.87%. Alternatively, if the convention was to use another instrument (as it is in different cap markets), say cash deposits, or deposit futures, then another forward calculation would be performed.

Step 2: Cap Volatility

Each of the caplets in the strip requires a volatility. The market convention is to quote the cap's on a "flat" vol basis, which is the single "summary" forward (in this example LIBOR) rate volatility that is used for each caplet. Chapter 20 provides additional volatility considerations, and Section 14.6.17.3 discusses some basic volatility issues between swaps, caps, and bond options. See [8.b] for a detailed treatment of IR volatility issues.

For this example, it will be assumed that this 18-month 3.90% cap is quoted in the markets (in units of "flat" yield vol) at 11.55/11.93% (bid/offer). So, assume you will need to "lift the offer" to complete the trade (i.e. you will be paying 11.93% vol for this cap).

However, market quoted spot vols could have been used also. For example, caplets are very closely related to options on deposit futures, such as options on EuroDollar futures. If the caplets' dates and other parameters match that of those listed options, then the vols from each of corresponding listed options could be used as spot vols[408] for the caplets. If the

[408] In fact, some small adjustments are required for a variety reasons, including so-called "convexity adjustments" introduced in Section 14.6.21 and detailed in [4.a] or [8.b]. Also, keep in mind that caplets are almost always European, while deposit futures options are almost always American, and may have other differences, such as "paid in arrears", etc., see below.