$$Call = e^{-qt} P N(d_1) - e^{-rt} K N(d_2) \tag{16.10}$$

and

$$d_1 = \frac{\ln\left(\frac{P}{K}\right) + \left(\sigma^2 / 2\right)t}{\sigma\sqrt{t}}, \quad and \quad d_2 = d_1 - \sigma\sqrt{t}$$

but where

$$\sigma = \frac{\sigma_{NormalBSM}}{\sqrt{3}}$$

> Caveat: This approach is almost surely NOT suitable for any real trading/risk management. However, it does provide a "cheap & cheerful" approximation to the valuation, and can also server as a "tool" in more sophisticated methods (e.g. certain variance reduction methods for Monte Carlo simulations etc., see Chapter 19 for an introduction, and [3.a] and [9.a] for details).

Discrete Averaging Options Pricing

Many AAveP Asians are contracts relying on discrete sampling of the average (e.g. daily, monthly, etc.). Continuous averaging methods are not acceptable for these cases unless the averaging frequency is very high (at least daily, as illustrated in Section 16.1.2.2). One of the common methods is another derivation by Levy [76] that produces a valuation for Asian options with discrete averaging. This is also a fairly well known formulation, and it too inherits all the usual baggage of shoehorning an averaging (Normal) process into a risk-neutral/arbitrage-free (Log-Normal) BSM setting. This approach is constructed by specifying the total number of (averaging) intervals in the option's life, n_{Fix}. It is also allows for a history of prices taken on the appropriate sampling dates prior to the current/valuation date , t_{Now}, called $P_{Hist}[1:m_{Fix}]$, where m_{Fix} is the last fixing date/point just prior to t_{Now}. If t_{Now} is the first day of an Asian, and it is a normal[457] Asian, then $n_{Now} = 0$, and $P_{Hist}(n_{Now})$ is a vector of length 1, and containing just today's (now's) price. Then:

$$Call_{AAveP,Disc} = e^{-rT}\left(E_1\, N(d_1) - K_{Pavd}\, N(d_2)\right) \tag{16.11}$$

with $E_1 = E\left[A(T)\right]$ being the expectation (the 1st moment) of the average price:

$$E_1 = \frac{Pe_1(1-e_2)}{(1-e_3)(n_{Fix}+1)}, \text{ or } E_1 = \frac{Pe_5(1-e_4)}{(1-e_3)(n_{Fix}+1)} \text{ when } t_{Now} < t_0$$

[457] Some Asians are defined to include a history of prices prior to the inception date, and those would be a departure from "normal" or "vanilla" Asians. Though, clearly, if a normal Asian was already on the books, then P_{Hist} would be the relevant price history from inception date to now, see [9.a] for detailed treatment.

where

$P = P_{Hist}\left(n_{Now}\right)$, note: this formulation includes the first/inception date in the averaging,

$g_0 = r - q$

$$K_{Pavd} = K - P_{Ave}\frac{\left(m_{Fix}+1\right)}{\left(n_{Fix}+1\right)}$$

$e_1 = e^{g_0(1-\xi)dt_{Fix}}$, $e_2 = e^{g_0(n_{Fix}-m_{Fix})dt_{Fix}}$, $e_3 = e^{g_0 dt_{Fix}}$

$e_4 = e^{g_0(n_{Fix}+1)dt_{Fix}}$, $e_5 = e^{-g_0(t_0-t_{Now})}$

$dt_{Fix} = \frac{AFreq}{DPY}$ = the (uniform) time interval between fixings (Levy's *h*).

$m_{Fix} = \frac{t_{now}}{dt_{Fix}} - Mod\left[\frac{t_{now}}{dt_{Fix}},1\right]$ = the number of fixing intervals already taken place.

$n_{Fix} = \frac{t_n}{dt_{Fix}} + 1$ = the total number of fixing (uniform) fixing intervals in the entire option's life (this assumes the inception date of the option is also a fixing date (otherwise remove the "+1").

$\xi = Mod\left[\frac{t_{now}}{dt_{Fix}}, dt_{Fix}\right]$ = the portion of the fixing interval since last fixing (i.e. the total year fraction since last fixing is = $h*\xi$. This variable is referred to as "xi" in the accompanying code).

> Aside: These variable definitions are "natural" for mathematical developments, but may not be the most convenient for real trading. Levy's formulations require first setting the total number of averaging intervals in the option's life. For example, a monthly averaging 1.25 year Asian would require setting n_{Fix} to be 15 as the calculation's input. In real trading, it is often more convenient to specify the number of days (our hours, or whatever) in the sampling periods. For example, assuming the units are days/interval, all the trader would do for the 1.25 year monthly Asian is input 1/12 (without having to calculate the number of intervals etc.). Converting Levy's format above to something a bit more practical is not rocket science, but it is a bit tedious (as is often case when adapting theory to reality[458]). The accompanying software provides Levy's method re-based with a few "real world friendly" alterations. Details for such alterations for exotics are provided in [9].

[458] As is often found, real world implementations are dominated by code that can determine if it's a Wednesday or a holiday, and the vast amount of real world minutia, rather than stochastic calculus.

and

$$K_{Pavd} = K - P_{avd} = K - \frac{P_{Ave}\left(m_{Fix}+1\right)}{n_{Fix}}, \text{ where } P_{Ave} = Ave\left[P_{Hist}\left(1:n_{Fix}\right)\right] = \frac{1}{n_{Fix}}\left(\sum_{i=1}^{n_{Fix}} P_i\right)$$

where

$$d_1 = \frac{\frac{\ln\left(E_2\right)}{2} - \ln\left(K_{Pavd}\right)}{v_0} \text{ and } d_2 = d_1 - v_0$$

with $v_0 = \sqrt{\ln\left(E_2\right) - 2\ln\left(E_1\right)}$

and $E_2 = E\left[A\left(T\right)^2\right]$ the expectation (the 2nd moment) of the average price as:

$$E_2 = \frac{P^2 e_{2X}\left(a_1 - a_2 + a_3 - a_4\right)}{\left(n_{Fix}+1\right)^2}, \text{ or } E_2 = \frac{P^2 e_{2Y}\left(b_1 - b_2 + b_3 - b_4\right)}{\left(n_{Fix}+1\right)^2} \text{ when } t_{Now} < t_0$$

with

$$e_{2X} = e^{-2\xi\left(g_0 + \frac{\sigma^2}{2}\right)h}$$

$$a_1 = \frac{e^{\left(2g_0+\sigma^2\right)h} - e^{\left(2g_0+\sigma^2\right)h\left(n_{Fix}-m_{Fix}+1\right)}}{d_1}, \quad a_2 = \frac{e^{\left(g_0\left(n_{Fix}-m_{Fix}+2\right)+\sigma^2\right)h} - e^{\left(2g_0+\sigma^2\right)h\left(n_{Fix}-m_{Fix}+1\right)}}{d_2}$$

$$a_3 = \frac{e^{\left(3g_0+\sigma^2\right)h} - e^{\left(g_0\left(n_{Fix}-m_{Fix}+2\right)+\sigma^2\right)h}}{d_3}, \quad a_4 = \frac{e^{2\left(2g_0+\sigma^2\right)h} - e^{\left(2g_0+\sigma^2\right)h\left(n_{Fix}-m_{Fix}+1\right)}}{d_4}$$

with

$$d_1 = \left(1-e^{g_0 h}\right)\left(1-e^{\left(2g_0+\sigma^2\right)h}\right), \quad d_2 = \left(1-e^{g_0 h}\right)\left(1-e^{\left(g_0+\sigma^2\right)h}\right), \ d_3 = d_2, \text{ and}$$

$$d_4 = \left(1-e^{\left(2g_0+\sigma^2\right)h}\right)^2$$

and when $t_{Now} < t_0$

$$e_{2Y} = e^{\left(2g_0+\sigma^2\right)\left(t_0-t_{Now}\right)}$$

$$b_1 = \frac{1-e^{\left(2g_0+\sigma^2\right)h\left(n_{Fix}+1\right)}}{d_1}, \quad b_2 = \frac{e^{g_0\left(n_{Fix}+1\right)h} - e^{\left(2g_0+\sigma^2\right)h\left(n_{Fix}+1\right)}}{d_2}$$

$$b_3 = \frac{e^{g_0 h} - e^{g_0 (n_{Fix}+1) h}}{d_3}, \quad b_4 = \frac{e^{(2g_0+\sigma^2)h} - e^{(2g_0+\sigma^2)h(n_{Fix}+1)}}{d_4}$$

A formulation for AAveP puts could be arrived at by put-call parity, but Levy has worked out an explicit put pricing analogue, as:

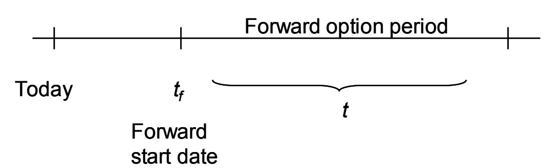
$$Put_{AAveP,Disc} = e^{-rT}\left(E_1\, N(d_1 - 1) - K_{Pavd}\, N(d_2 - 1)\right) \tag{16.12}$$

> Aside: Recall that with Log-Normal pricing the puts and calls had "multiplicative -1's" in the cumulative probability term adjustments, where as here (in the Normally distributed prices world of averages) the adjustments are "additive -1's".

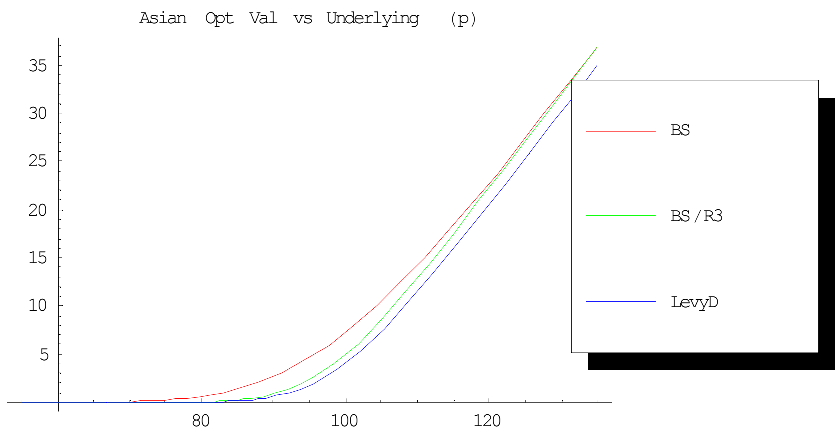
> Aside: As earlier, this is a "shoehorned" result and means that these types of formulation are approximation, even though they "appear" to be closed form solutions.

> Aside: Notice how much "messier" the formulas become when "discrete time" is involved. This is in part why academic/theoretical efforts "prefer" continuous time/continuous price. Methodologies based on numerical methods discrete based dynamics are, in a sense, easier.

These Levy 1992 [76] Asian option formulas also permit setting t_{Now} prior to the start date of the option (i.e. $t_{Now} < t_0$), illustrated in the schematic to the right. Circumstances, other than mark-to-market issues, that give rise to these types of valuations are discussed in [9].

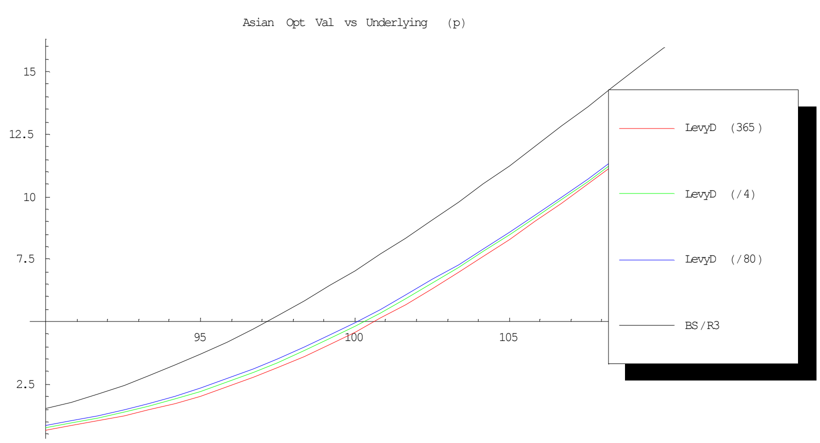


As an illustration of these formulas, consider the pay-out profile for a vanilla call, a continuous averaging Asian using the Root3 trick, and applying Levy's discrete Asian valuation formula in Equation (16.11). The image below shows these three results assuming the following inputs: $P = 100$, $P_{Hist}=\{100\}$, $K = 100$, $Vol = 15\%$, $T = 1$, $t_{now} = 0$, $r = 6\%$, $q = 0\%$, $DaysInAveInterval = 365$ (i.e. a single averaging period since it is a 1-year spot start option), and $DPY = 365$.



As expected, the Asian (blue) is much less "expensive" compared to a vanilla call (red), though of course it provides a different risk/return (there are no "free-lunches" in a risk-free/arbitrage-free BSM universe). The Root3 "trick" (green) valuation is sufficiently far away from the "better/proper" Asian (blue) valuation, though it is only for continuous averaging.

If the contract was re-specified for several discrete averaging periods *DaysInAveInterval* = = 365 (red), 365/4 (green), 365/80 (blue) (i.e. increasing averaging frequency with decreasing length of averaging periods), then the image below illustrates that the Root3 trick is likely not sufficiently close for some approximations, as the high frequency discrete averaging (blue) is essentially the discrete "converged" to "continuous" averaging, and the Root3 (black) is still quite far away (many full points of options premium). This image has been magnified compared to the image above since the impact of averaging frequency here is quite benign. Some additional implications of sampling frequency are provided in Section 16.1.2.2.



Of course, this is all assuming that this Levy formulation is in fact the "correct" result. As shown in [9] with numerical methods, the closed form approximations such as Levy's and