7.7.3 A First Model for the Time Evolution of Uncertainty

The story so far: this "first model" of uncertainty provides expectations of outcomes for one future period (i.e. for one time step). This may be sufficiently complete for some problems (e.g. a European Option with no rebalancing, no interim cash flows etc). If that is the case, then, this first model is ready for "P&L verification", and further development as in Chapter 22.

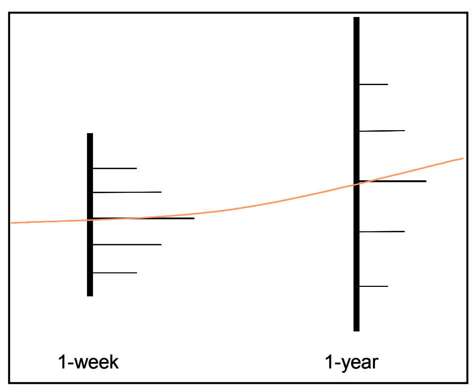
However, in many important real world situations multiple forward dates are significant[102]. For example, there may be multiple rebalance dates (might be daily, or more frequent), multiple cash flow dates, or multiple "event" dates (e.g. early or partial exercise in a Bermudian or American option). Even when there is only a single forward date, the model benefits from a generalisation, so that it may be used with greater flexibility.

So to complete this "first model of uncertainty", the model must be extended to permit accounting for multiple future dates, or the generalisation to variable term to forward dates.

As above, the development relies on both technical considerations and intuition.

7.7.3.1 An Intuitive Approach to the Time Evolution of Uncertainty

One intuitive approach is the "voting game" seen above in Section 7.7.1.1. Votes of the "audience's expectation" of the forward price of the S&P were used to see how traders as a group "see" the S&P distribution 1-week forward. Suppose that this game was played again, but this time the question was for S&P expectations 1-year forwarded. The results of this "1-yr voting game" are illustrated in the image to right, together with the earlier "1-week voting game" for expectation of the S&P distributions.



Again, the 1-year distribution is "fat in the middle" and "thin in the tails" (though again this does not, in itself, mean that the distribution is necessarily Gaussian/Normal). However, the average of the 1-year distribution is higher than the 1-week case (reflecting the audience's view that equity markets tend to drift upwards). Importantly, the distribution is also wider (reflecting the audience's increased uncertainty with increasing length of forecast horizon).

[102] Again, there is also the issue of continuous time models, wherein even a "single time step" to a single forward date still constitutes an infinite number of infinitesimal steps.

Does this mean that each forward date will require a separate model/distribution derivation? Alternatively, is it possible to make use of the observation that the shapes of the forward distributions are somehow similar?

One possible simplification is to suggest that the distribution on any given forward date is from the same "family of distributions", but perhaps with different parameters (e.g. averages, standard deviations, etc). This would considerably simplify the model, compared to the general case where each forward date is permitted to have a completely independent distribution. This simplification permits that the "basic shape" of choice is "shifted" and "stretched" over time as required, so that the "shape evolves over time" as required. For example, Figure 7.7 – 3 illustrates two possible continuums of forward distributions. There are two different distributions, with two different types of "evolutions" shown to remind us that there are many choices of specific shape of uncertainty, and also there are choices for the nature of the time evolution[103].

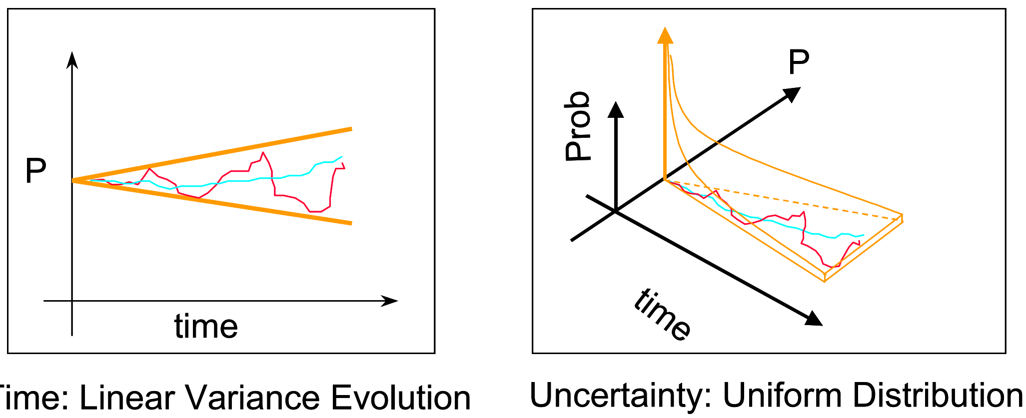


Figure 7.7 – 3 a) A Uniform distribution, with "linear" evolution over time forms this "shrinking box". The image on the left shows two possible forward price paths as seen from above. Notice that with this choice, the forward paths are bounded by the "triangular uncertainty envelope".

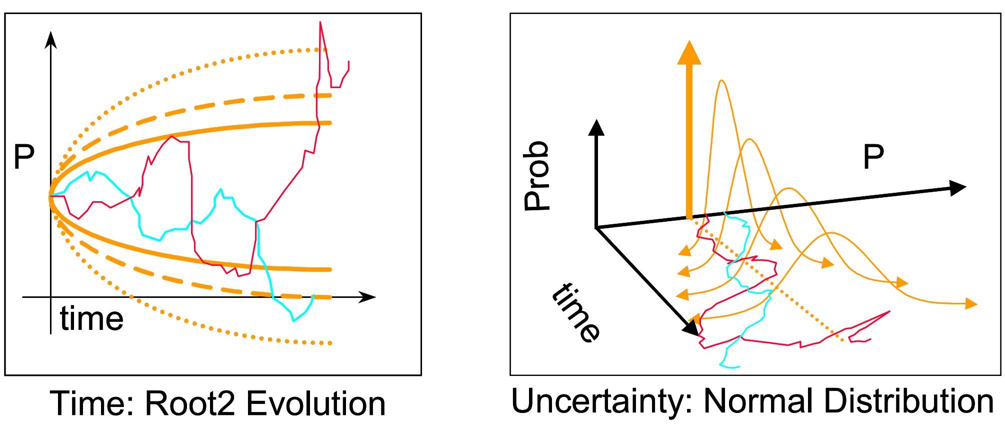


Figure 7.7 – 3 b) A Gaussian (Normal) distribution, with "root-2" evolution over time (i.e. the uncertainty increases as the square root of time) forms this "Mountain Range". The image on the left shows two possible forward price paths as seen from above. There are 3 sets of dashed lines depicting the 1-, 2-, and 3-standard deviation "envelopes", which follow a $\sqrt{t}$ shape.

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[103] As always, P&L verification is the trader's ultimate test of "goodness".

This type of "mountain range" is embedded in all valuation processes that rely on uncertainty. The simplification of choosing each forward distribution to be from the same "family" of shapes is seen in virtually all theoretical and market convention calculators, though these mountain ranges are sometimes "buried" in the calculator and so it may not be immediately obvious (but – they are there). Indeed, virtually all market convention and theoretical models begin with the Gaussian/Root-2 uncertainty/evolution.

How do the market convention methods "know" if the Root-2 evolution is correct? Without backtesting for P&L impact – they don't. However, intuition suggests that it is consistent with expectations of forwards as reflected in the "voting" process above (for the S&P index at least). Chapter 21, and the Vol 2 "trading" book [8.a], provide additional verification and backtesting, and the next Section considers some technical issues as well.

7.7.3.2 Some Technical Considerations for a 1st Model of the Evolution of Uncertainty

One particularly compelling technical consideration for the choice of the "evolution of uncertainty" is (a bit like in Section 7.7.1.1) that if the evolution is Root-2, then the mathematics is very much easier for the "quant guys". Choices other than Root-2 are very much more difficulty, or even (mathematically) intractable. So, this "confinement", combined with the "intuitive" result from above leads to accepting Root-2 as sufficiently close and good enough for a first model for this "time evolution component".

This is indeed what happens in practice, and the formula is generally not tested very carefully. It will be seen in Chapter 11, and also in [8.a], that this type of formulation may be acceptable under some circumstances. However, there are markets, and also important derivative and exotic structures, wherein these "first model assumptions" are very bad approximations. Nevertheless, for now, assume that this first model is good enough (the market convention is built on this assumption as well), and later in Chapter 22 and [2] where the question of Model Arbitrage will be tested further.

7.7.3.3 A First Formulation of the Evolution of Uncertainty

Based on the intuition and technical considerations from above, the choice for a formulation of the evolution of uncertainty is one that essentially "stretches" a single "base" shape or distribution by an amount that is proportional to the elapsed time to the forward dates of interest.