9.4.3 Gamma

Gamma is the instantaneous sensitivity of an option's Delta to an infinitesimal change in the underlying price or index, with all other parameters held constant. Historically, this sensitivity has been represented by the Greek letter γ or its uppercase version Γ. Mathematically, the formula for Gamma is arrived it by taking the (partial) differential of the option's Delta formula wrt the underlying price, *P*. Equivalently, this is the second partial derivative of the option's price wrt the underlying price *P*. Here, only the final results is presented[234], which for the option pricing formula in Equation (8.44) is:

$$\begin{aligned} Gamma\big|_P &= \frac{\partial^2}{\partial P^2}\left(e^{-qt}PN\left(d_1\right)-e^{-rt}KN\left(d_2\right)\right)=\frac{\partial}{\partial P}Delta\big|_P \\ &= e^{-qt}\frac{N'\left(d_1\right)}{P\sigma\sqrt{t}}=e^{-qt}\frac{\frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}}}{P\sigma\sqrt{t}}=\frac{e^{-qt}e^{\frac{-d_1^2}{2}}}{P\sigma\sqrt{t2\pi}} \end{aligned} \tag{9.11}$$

where the derivative of the Cumulative Normal Distribution, *N'()*, is:

$$N'\left(d_1\right)=\frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}} \tag{9.12}$$

So,

$$Gamma\big|_P = \frac{e^{-\left(\frac{d_1^2}{2}+qt\right)}}{P\sigma\sqrt{t2\pi}} \tag{9.13}$$

where all the elements have their usual meaning.

Gamma (for vanilla options) is the same for calls and puts that are otherwise identical.

It can be shown (see [3] for details) that for a wide range of conditions (e.g. there are no "kinks" or "gaps" etc in the pay-out profile), that an empirical sensitivity will converge to the instantaneous one, as:

$$Gamma\big|_P \xrightarrow[\Delta P\to 0]{Limit} \frac{Delta\big|_{P+\Delta P}-Delta\big|_P}{\Delta P} \tag{9.14}$$

On substituting for Delta, one arrives at:

$$Gamma\big|_P \xrightarrow[\Delta P\to 0]{Limit} \frac{\frac{\left(V\big|_{P+\Delta P}-V_P\right)}{\Delta P}-\frac{\left(V_P+V\big|_{P-\Delta P}\right)}{\Delta P}}{\Delta P} \tag{9.15}$$

or

[234] Rules for differentiation are provided in [3] and elsewhere.

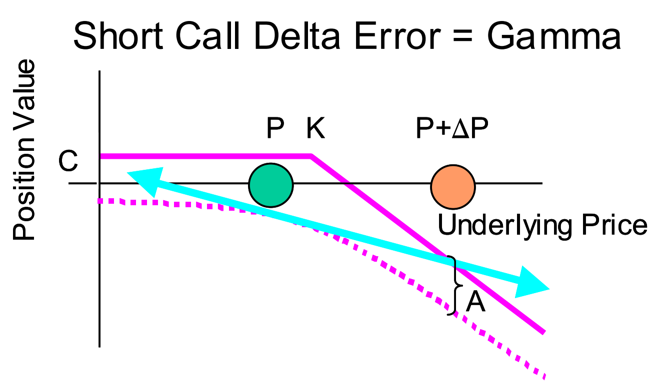
$$Gamma\Big|_P \xrightarrow[\Delta P \to 0]{Limit} \frac{V\big|_{P+\Delta P} - 2V_P + V\big|_{P-\Delta P}}{\Delta P^2} \quad (9.16)$$ [235]

> The traders in the audience will recognise the numerator as the standard "butterfly trade". This is not exactly the "butterfly" of Chapter 4.8.3. This is a standard construction for "trading curvature", such as the curvatures of yield curves, vol curves, vol skews etc. The trade is long one contract of the "higher point", go short two contracts of the current/middle point, and long a contract of the "lower point" For example, long a 2-year bond, short 2 10-year bonds, and long a 30-year bond would be a 2's/10's/30's butterfly that would be used when the yield curve has unsustainable "humps". A similar construct is used to arbitrage "humps" in vol curves, and vol skews (see [2] and [8.c]).

That is, if an option is priced for three different values of the underlying price *P* where the difference in those prices is ΔP then the curvature $\frac{V\big|_{P+\Delta P} - 2V_P + V\big|_{P-\Delta P}}{\Delta P^2}$ will become (exactly) Gamma as ΔP approaches zero.

Mathematically, Gamma is simply the curvature (slope of slope) of the option's pay-out/value profile taken at a specific underlying price *P*. Technically, curvature can also be referred to as "convexity". Indeed, in bond trading convexity serves (sort of[236]) the same purpose for measuring bond curvature as Gamma does for options. Unfortunately, the word convexity is used for a variety of other trading issues, and it may be best to restrict Gamma to this particular curvature.

From one trading perspective, Gamma is the amount by which Delta at *P* will be "in error" when the price moves to *P+ΔP*. The image to the right illustrates that if the underlying price moves up to *P+ΔP*., then Delta will imply/project a position value that is "in error" by an amount A, due to the curvature (Gamma).



From another trading perspective,
curvature is what the long is paying for, and what the short is selling.

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[235] In fact, there are other constructs that can be used to create approximations to the 2nd derivative via Taylor's methods, as detailed in [3].

[236] As per earlier, bond valuation formulas relate the bond's price to its IRR, and the convexity there is between bond price and IRR. This is not quite the same relationship as with options, since Gamma is in connection wit underlying price, rather than wrt to another measure of value. A second order partial of options value wrt volatility (sometimes called Vomma or Vega convexity) would be a close match to bond convexity.

Long (vanilla) calls and puts have positive Gamma, and shorts have negative Gamma. The "flatter" Delta becomes, the less curvature there is. Thus deep ITM and deep OTM options have vanishing Gammas.

For vanilla options, the Gamma is the same for otherwise identical calls and puts, and this shows up also in the PCP from Chapter 8. For example, long a call and short a put synthesises the underlying, which has zero (Gamma) curvature, since Gammas of the call and put cancel exactly.

9.4.3.1 Example: A Vanilla Call Gamma

Using the previous example option, with the option struck at 103, when the current (spot) market price is 100, with funding rate of 6% (continuously compounded, annualised), no income yield, 1-year to expiration, and a standard deviation of returns of 18% "annualised", the Gamma is:

$$d_1 = \frac{\ln\left(\frac{P}{K}\right) + \left(r - q + \sigma^2/2\right)t}{\sigma\sqrt{t}} = \frac{\ln\left(\frac{100}{103}\right) + \left(0.06 - 0 + (0.18)^2/2\right)1}{0.18\sqrt{1}} = 0.259117765$$

so

$$Gamma\big|_P = \frac{e^{-\left(\frac{d_1^2}{2} + qt\right)}}{P\sigma\sqrt{t2\pi}} = \frac{e^{-\left(\frac{0.259117765^2}{2} + 0\right)}}{100 * 0.18\sqrt{1*2*3.1415}} = 0.021432077$$

This may be interpreted to mean any of the following:

- The "curvature" of the pay-out profile wrt the underlying price is 0.02143
- The "projection" or "prediction" error of Delta over a distance ΔP will require a correction proportional to Gamma (see below and Chapter 10).
- If it was required to reduce or eliminate the Gamma curvature risk, then it is this value of Gamma that must "matched" by an off-setting position. The offsetting position could be in a single option with this amount of Gamma, or multiple contracts of options with combined Gamma that is negative to this target Gamma.

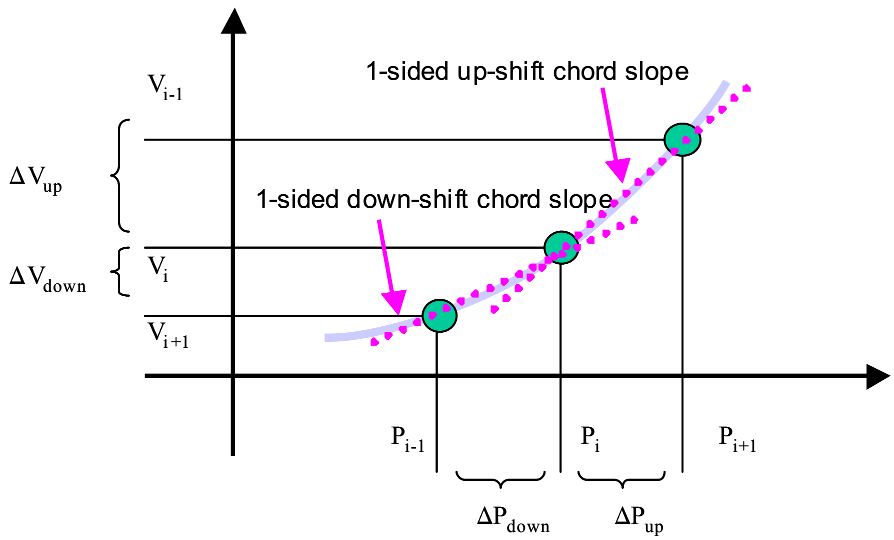
9.4.3.2 Example 2: A Vanilla Call Empirical Gamma and V01

As the up and down shifted values of the option were already calculated for the empirical Delta approximation above, it is now a simple matter to obtain an empirical approximation for Gamma:

$$Gamma|_P \cong \frac{V|_{P+\Delta P} - 2V_P + V|_{P-\Delta P}}{\Delta P^2} = \frac{8.669402963 - 2*8.663379618 + 8.657358417}{0.01^2}$$
$$\cong 0.021439262$$

This differs from the analytic solution by 7.18543E-06, or a relative difference of 0.03353%.

The image below shows that the empirical Gamma is just the slope of two chord slopes (i.e. the slope of the two one-sided empirical Deltas).



As will be shown in Chapter 10, the implication for "predicting" the error in the change in the value of the option using Delta alone due to a movement in *P* is proportional to Gamma as $\frac{\Delta P^2}{2} Gamma\Big|_P$. Thus, relying only on Delta produces

$$Value|_{P+.01,Delta} \cong V + \Delta P * Delta$$

so

$$Value|_{P+.01,Delta} \cong 8.663379618 + .01 * 0.602227755$$
$$\cong 8.66940190$$

This has an error compared to the actual result of:

$$Diff\left(Value|_{P+.01,Delta}\right) = 8.66940190 - 8.663379618$$
$$= 1.06769\text{E-}06$$

This may not seem like a big error, but this is for 0.1 bps on a notional of 100. If the position notional was, say, 100 million (a relatively small options position in professional trading), and the market moved 1%, then the error/slippage would be on the order of 10,000 (e.g. dollars). Since a 1% market movement can happen every day (indeed more than once per day), the risk measurement implications are profound.

By contrast, correcting the prediction for the effect of Gamma is as:

$$Value\Big|_{P+.01, Delta+Gamma} \cong V + \Delta P * Delta + \frac{\Delta P^2}{2} Gamma$$

or

$$Value\Big|_{P+.01, Delta+Gamma} \cong 8.663379618 + .01 * 0.602227755 + \frac{0.01^2}{2} * 0.021432077$$
$$\cong 8.669402967$$

This implies a projection error of -3.91286E-09, which is about 1,000 times smaller compared to the Delta only projections. That is, for the 100 million notional 1% market movement case, the tracking error drops from about 10,000 (dollars), to about 3 (dollars).

Though note that this type of projection is only appropriate for monotonic portfolios, as will be demonstrated in Chapter 10.

9.4.3.3 Some Properties of Gamma

Figure 9.4 – 3 illustrates both the call option's value and its Gamma (green) at expiration (a), and 1-year to expiration (b). The expiration (a) image shows that Gamma is singular (goes off to infinity) at the strike, but is zero elsewhere. This is to be expected since at expiration the pay-out is "constant" (the Delta is "flat") everywhere except at the strike, where there is a "kink" (singularity). Strictly speaking, this is slightly different type of singularity compared that in the Delta profile, and is much more difficult to "manage".

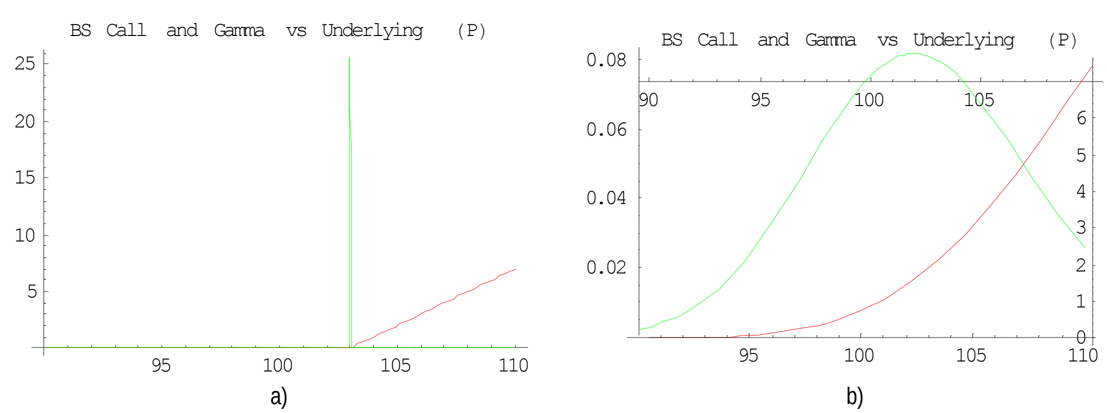


Figure 9.4 – 3. a) Call value and its Gamma (green) at expiration. b) Call value and its Gamma (green) 1-year to expiration.

Prior to expiration shown in image (b), Gamma has a (finite) maximum at the strike. This further demonstrates that "maximum (time) value" of the option is ATM.

Figure 9.4 – 4 illustrates the option's Gamma varying simultaneously with underlying price and time to expiration. The peak is the singularity at the strike on expiration day.

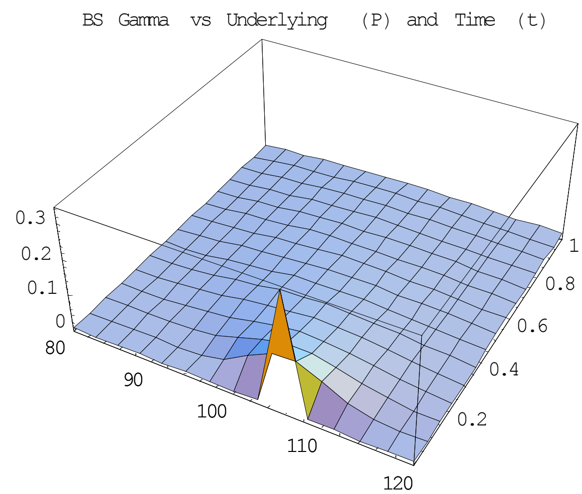


Figure 9.4 – 4. Option Gamma over a range of time and underlying price.

As with Delta, a moment before expiration, if the market is on either side of the strike, a tiny movement in the market will have a profound impact on the Gamma.

9.4.4 Vega

Vega is the instantaneous sensitivity of an option's price/value to an infinitesimal change in the input volatility (i.e. standard deviation of returns in the vanilla case), with all other parameters held constant. Historically, this sensitivity has been represented by the letter υ, as there is not actually a Greek letter called Vega.. Mathematically, the formula for Vega is arrived it by taking the (partial) differential of the option's pricing formula wrt to volatility (standard deviation of returns), σ. Here, only the final results is presented[237], which for the option pricing formula in Equation (8.44) is:

$$\begin{aligned} Vega\big|_{\sigma} &= \frac{\partial}{\partial \sigma}\left(e^{-qt}PN\left(d_1\right)-e^{-rt}KN\left(d_2\right)\right) \\ &= e^{-qt}P\sqrt{t}N'\left(d_1\right) \end{aligned} \tag{9.17}$$

where t is time to expiration, or

[237] Rules for differentiation are provided in [3] and elsewhere.