13.1 Stochastic Differential … What?

This and the next Sections review some definitions and explain why there is a need for "yet another calculus".

Although it is the distance of a journey or the balance in a bank account that is ultimately important, the process for "getting there" is most easily handled by "managing speed" (or return). Thus, it is the rate of change (speed or return) that is the quantity of immediate interest from a modelling perspective.

13.1.1 Differential Equations

A model that expresses a process in terms of changes or difference is a "difference" equation, such as forecasting the change in the forward price with a drift plus an uncertain term as:

$$\Delta P = Pr\Delta t + P\Delta z \tag{13.1}$$

where the first term is the change due to drift, and the second term is change due uncertainty.

If this model is to be expressed in continuous time/price, and if the equation and its variables conform to the requirements of the "usual" (Riemann) calculus from Chapter 3, then it can be written as

$$dP = P\,rdt + Pdz \tag{13.2}$$

which is now a differential equation, rather than a difference equation.

The process of converting the differences to differentials required rules of calculus.

Since continuous time models are far more convenient form a mathematical perspective, there is a need to develop a framework (a calculus) for differential equations as models of price/returns dynamics.

13.1.2 Why Another Calculus?

A calculus is just a set of rules for dealing with quantities expressed as speeds or changes (i.e. slopes). Since slopes only exist under certain conditions the rules of (any one) calculus are only applicable under those conditions.

The conversion of Equation (13.1) into the differential form Equation (13.2) cannot be performed correctly using the "usual" calculus if the uncertain term z is stochastic. This can be seen by reviewing the requirements for the applicability of Riemann calculus reviewed in Chapter 3.10.

In particular, a stochastic process has (at least) two properties that violate those requirements:

- It is not C^1 continuous
- It reveals additional information on magnification.

Figure 13.1– 1 shows that the any attempt to "take a slope in the limit" by the "usually" shrinking of the chord slope is fraught with difficulties since the slope fluctuates wildly as the interval shortens.

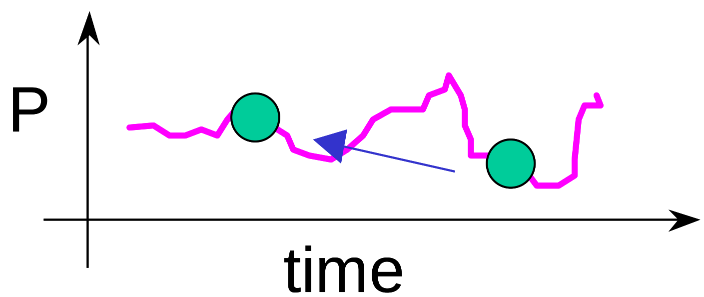

Figure 13.1– 1 The repeated magnification of a stochastic function will reveal an infinite level of

Figure 13.1– 2 shows that the situation is even more complicated since on magnification, a stochastic process will "reveal" new information.

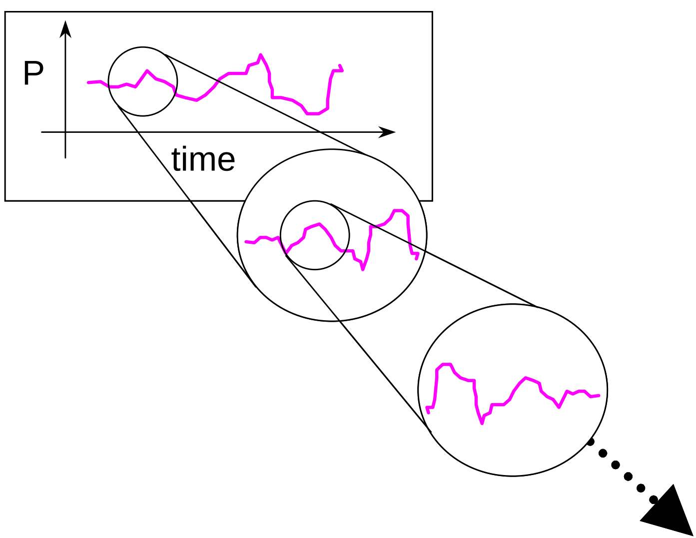

Figure 13.1– 2 The repeated magnification of a stochastic function will reveal an infinite level of detail and each of the new "details" will be newly random on each magnification.

Therefore, shrinking the chord slope will not only fluctuate wildly, but also will not converge to a unique answer.

This is in sharp contrast to the usual calculus where well-behaved functions reveal decreasing information on magnification (converging to a straight-line in the limit).

As such, the "usual" calculus is not applicable, and so stochastic problems require a different set of rules for differentiation.

So what? Why should I care if the "usual" calculus is not applicable to my forecasting machine? Why not just use the equations from Chapter 12 and make my forecasts on that basis?

That is certainly possible. However, there are a number of situations of immediate interest to traders where a new calculus is critical.

- Position keeping/risk management: The slope or the change in a positions value is the primary representation of risk and the key input into rebalancing calculations. As such, a "correct" slope calculation is required there.
- Options/derivatives trading: it is convenient (and often critical) to be able to relate the value of a derivative to changes in the value of the underlying instrument. These "changes" are stochastic slopes, and so require a stochastic calculus.
- Market making: the risk neutral valuation methodology at the base of all market convention market making in options/derivatives relies critically on relating slopes to pay-outs.