17.5 Finite Difference Methods

Thus far, the solutions methods have focused on the Differential Equation (DE) that models market dynamics, and which then superimposed another layer of “model” to account for instrument specific or trading specific issues. That is, the solution methods have only dealt with DEs of the form.

$$dP = \underbrace{P(r - q - \frac{\sigma^2}{2})dt}_{drift} + \underbrace{P\sigma_r dz}_{uncertainty} \qquad (17.57)$$

resulting in a simulation of forward prices/rates. Then, the remaining aspects of the pricing/hedging calculations were performed separately.

However, it has been seen Chapters 12 – 14 that “complete” instrument-pricing expressions can be derived by risk neutral/no-arbitrage methods, for example. Those derivations lead to (instrument/position) model DEs of the form:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial p^2}(\sigma p)^2 + \frac{\partial f}{\partial p}(r - q)p = rf \qquad (17.58)$$

This PDE is rather more complicated to solve, and the MC and Tree methods are not well suited to handle this type of PDE.

Instead, there are methods that deal directly with PDEs. The Finite Difference (FD) method is one such technique. FD can still produce solutions to just the underlying (market simulation) problem, and superimpose instrument and other factor as required. However, they can also deal directly with PDEs such as (17.58).

There are a number of advantages to PDE methods. Primarily, they can deal with very great complexity and high-dimensional problems in much more effective and manageable manner than the pure simulation-like (MC and Tree) methods. They also execute very much more quickly, and so can produce answers to complex problems with less computation effort.

The price for this robust and fast performance is that the implementation must be by those “who know what they are doing”, and even then, implementation can be a big task. Though, once a reasonably general FD engine has been developed, extending it to include additional complexities is usually relatively easy.

PDE methods such as FD (and Finite Element and Boundary Methods in Section 17.6) "break up the problem" into smaller "pieces" and then solve the (simpler, but many) approximations on those smaller pieces. Once done, they reconstruct the global solution from the many local solution from the solution of the many simpler problems.

Importantly, PDE methods can "break up" the problem, roughly speaking, to different degrees. One possibility is to use PDE methods to convert the PDE to a system of Ordinary Differential Equations (ODEs), solve the ODEs' and reverse out the PDE global answer. This has the advantage of preserving "more of the original PDE", but requires expert skill for solving the ODEs.

Often, the PDE methods are used to break up the problem further, and convert the PDE into a system of algebraic equations. Algebraic equations are relatively easy to solve, but there will be very many of them, thus requiring greater computational effort (than ODE methods).

Here, only the PDE to algebraic conversion is considered.

17.5.1 The Basic Idea

The essence of the FD technique is to approximate the differentials in the PDE as discrete slopes, thereby converting the PDE in to a system of algebraic equations. The resulting system of algebraic equations may be solved by any of the well-known methods[247].

In order to illustrate the motivation for the FD method, consider as an example, the Black-Scholes PDE:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -\frac{\partial V}{\partial t} \qquad (17.59)$$

where,

V value of option

S price of underlying instrument

σ volatility of S

r risk free rate of interest

This expression relates the change in the value of an option (V) to changes partially in the underlying instrument (S) and to changes partially in time (t). For example, the term

[247]However, for large problems implementation of the linear equation solving technology can be task unto itself.

$rS\frac{\partial V}{\partial S}$ is the rate of change (slope) of the option value with respect to an infinitesimal change in S. This term expresses the contribution of the price trend to the option price (i.e. the effect of drift). The term $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ provides the contribution of the random movements of price (diffusion or uncertainty). Finally, the term $\frac{\partial V}{\partial t}$ indicates that V also changes with the passage of time.

Pictorially, the term $\frac{\partial V}{\partial S}$ is the *tangent* or *slope* (i.e. the straight line) illustrated in Figure 17.5 – 1 a). This tangent slope can be approximated as illustrated in Figure 17.5 – 1 b) with the chord slope. This process of reducing a domain into segments is called discretisation. For many interesting cases, this approximation improves as the interval (ΔS) shrinks[248]. In these situations, the FD method can be used to approximate the solution. In particular, the entire range is broken up into small segments, wherein slopes are approximated as a series of straight lines[249]. Notice that the discrete approximation in Figure 17.5 – 1 b) is slightly different from the exact slope in Figure 17.5 – 1 a). This is an approximation error, which generally increases, as the segments are made wider.

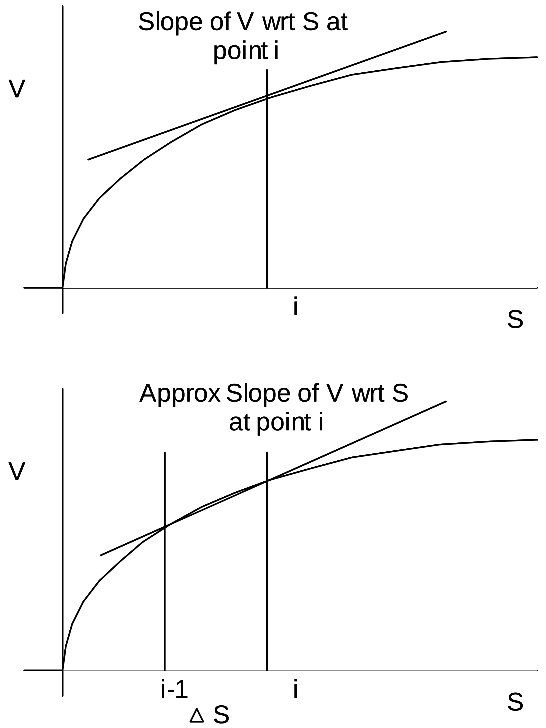

Figure 17.5 – 1. Tangent slope of a function and chord slope approximation.

[248]There are model problems for which this *convergence* fails. This is a topic outside the current scope.

[249] In fact, one is not restricted to linear segments; they can be as sophisticated as appropriate. The linear case is, however, the simplest and most common case.

Using this idea, the Black-Scholes PDE can be rewritten in approximate form as:

$$\frac{1}{2}\sigma^2 S^2 \frac{\Delta^2 V}{\Delta S^2} + rS\frac{\Delta V}{\Delta S} - rV \cong -\frac{\Delta V}{\Delta t} \qquad (17.60)$$

where the Δ terms imply a small change (i.e. arithmetic difference). For example, the term $\frac{\partial V}{\partial S}$ (now replaced by $\frac{\Delta V}{\Delta S}$) is evaluated as $\frac{(V_i - V_{i-1})}{(S_i - S_{i-1})}$ [250].

Now the original PDE is a "difference" equation, and the difference are "simply" approximated by, say, linear segments. Thus, this equation becomes "algebraic" (as opposed to differential).

This type of discretisation of the Black-Scholes PDE leads to approximations as illustrated in Figure 17.5 – 2 a) and 2 b). Figure 17.5 - 2 a) illustrates the discretisation of the underlying price and time domain. While Figure 17.5 - 2 b) depicts the solution surface made of approximating *tiles* or *plates*. These tiles are the equivalents to the straight-line segments in Figure 17.5 - 1 b). But now, tiles are required since the option price changes with both *S* and *t* (i.e. in two directions).

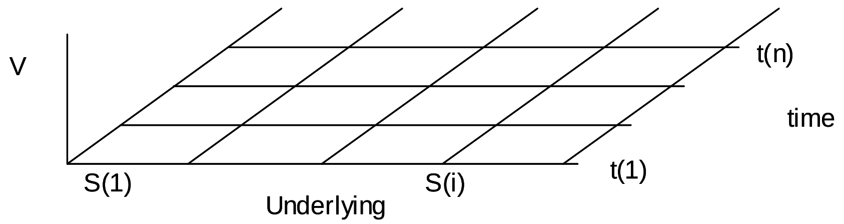

Figure 17.5 – 2 a) Discretisation in "space" (price) and time.

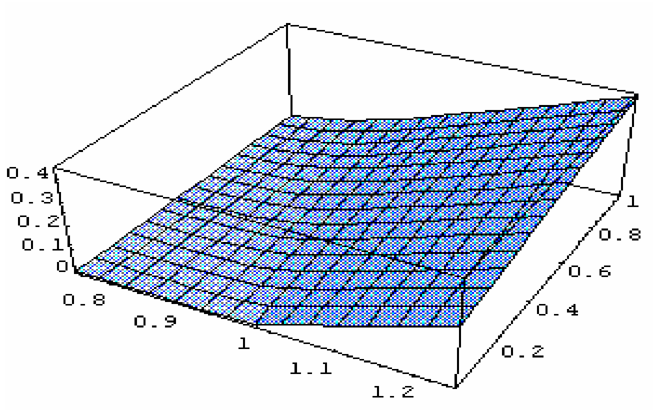

Figure 17.5 – 2 b) Solution of the differential options equation.

250The issue surrounding forward vs. backward differencing will be omitted from this section.