3.4 Higher Order Derivatives

Complex curves may support many "orders of slopes" or derivatives. The number of derivatives or orders of slopes of curves will vary from case to case, and possibly even from point to point along a curve.

In some cases, the highest order slope will simply be a "zero curve" (i.e. a flat curve with value zero, not to be confused with a zero-coupon curve etc). In other cases, the process of taking higher order derivatives will be terminated due to singularities (see Section 3.11). Keep in mind that slopes and derivatives apply "locally" (e.g. on interval or at a point), and so the process of differentiation may "terminate" differently at different points along the curve.

Higher orders derivatives correspond speeds of speeds, or higher accelerations (e.g. "accelerated accelerations[22]").

Figure 3.4 – 1 illustrates the function

$$y(x) = 3x^3 \qquad (3.19)$$

and its first three derivative functions ($\frac{dy}{dx} = 9x^2$, $\frac{d^2y}{dx^2} = 18x$, and $\frac{d^3y}{dx^3} = 18$)

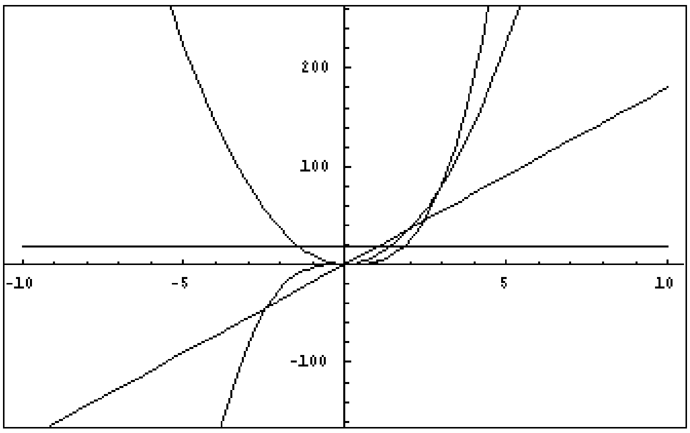

Figure 3.4 – 1. A first, second, and third slopes of a cubic curve

[22] Imagine you board at the rear doors on to the "tube" (a.k.a. subway, metro, etc). Suppose that you start running towards the front of the tube while the tube itself is accelerating up to speed. You are accelerating from standstill to a run simultaneously as the tube is accelerating, and so you experience "accelerated acceleration". In slope terms, your "experience" will exhibit at least a 3rd derivative.

In Figure 3.4 – 1, the higher order derivatives corresponded to "obvious extra features" in the curves. For example, in this case the higher order/more complex curves are non-monotonic.

This need not be so. There are curves that do not exhibit any obvious or "by inspection" special features, and yet may still posses higher or complex mathematical order or features[23].

$$
\begin{aligned}
& y(x) = e^{-rt} \\
& and \\
& \frac{dy}{dx} = -te^{-rt};\ \frac{d^2y}{dx^2} = t^2e^{-rt};\ \frac{d^3y}{dx^3} = -t^3e^{-rt}
\end{aligned}
\tag{3.20}
$$

Figure 3.4 – 2 shows the derivatives functions of the continuously compounding discounting factor, and that it supports many higher order derivatives, and yet "appears well behaved" and is monotonic everywhere.

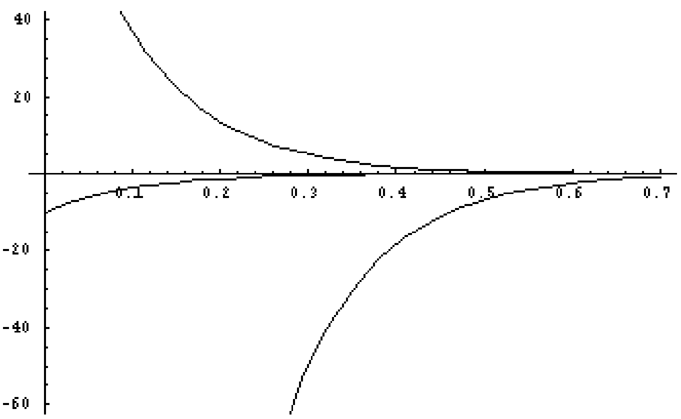

Figure 3.4 – 2. A first, second, and third slopes of an exponential curve

[23] See also Chapter 8.2.6 on Analyticity as another interesting illustration of this issue.

3.4.1 Higher Order Derivatives and "Fundamental" Properties

There are two important properties in the case that there are many higher order derivatives. One situation is that there are an infinite number of higher order derivatives (as opposed to a finite number). The other important consideration is the "convergence" properties of the higher order derivatives. That is, do the numerical values of the higher order derivatives diminish, or do they make a material contribution to the "value of the curve"?

For example, bonds do not exhibit too much curvature, and so Duration or V01 hedging is generally considered sufficient. Put differently, bond Convexity is so small in most cases that it's not worth the effort (e.g. bid/offer spread etc) to hedge.

In the case of options, however, there can be quite considerable curvature, and so the 2nd derivative (Gamma) is very important (has a significant value) in many cases[24]. However, it is rare to speak of hedging a 3rd derivative (whatever Greek letter that would have), since most (but not all) risk or pay-out profiles in trading are monotonic, and monotonic curves can be "risk managed", for the most part, with just the 1st and 2nd derivatives[25].

In the case the where there is significant non-monotonic behaviour there are also other methodologies available that obviate the need for (directly) using higher order derivatives. For example, Section 3.4.2 on "predictions" illustrates "scenario" analysis to be an alternate strategy to differentiation.

It cannot be emphasised that while slopes and derivatives are a most importantly tool in trading and risk management, sole reliance on them is likely to be "career limiting". Always be aware of the broader landscape of P&L and risk-adjusted P&L behaviour.

This entire notion of the convergence properties of higher derivatives is fundamental property of the curve (and so of the process described by the curve). It is closely linked with many important and advanced concepts in mathematics and trading. Chapter 8.1.3 on Taylor Series expansion considers this matter further. However, all of the modelling and curve fitting, and related material all contribute to this most important framework. For example, it is precisely due to these types of fundamental issues that process such as "price behaviour" in the markets are not amenable to modelling in an straightforward or easy manner (see also Chapter 14)

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[24] Notice that many Delta trades can "approximate" curvature, and so high frequency Delta (1st derivative) rebalancing can approximate Gamma (this is also at the basis of the Black-Scholes-Merton risk neutral methodology as shown in Chapters 11, 12, and 13). Alternatively, you could just hedge Gamma directly with another option. The various strategies of synthetic replication in trading require understanding of the calculus as well as the markets and instruments as introduced in [1], and detailed in the product books such as [8].

[25] Monotonic curves may have higher order derivatives, but their contribution is small compared with the frequency of rebalancing or other such interactions. Put differently, with frequent interaction, most "profiles" can be suitable "replicated locally" with just a 1st and 2nd derivative.

3.4.2 Slopes as Predictors vs. Alternate Methods

Slopes can be and are used in various interpolation and extrapolation contexts. In one sense, such results are "predictions". In the case of well-behaved curves[26], this type of prediction can be a very useful tool not only with "explicit predictions", but also as a proxy for explicit predictions (e.g. cheaper computationally).

This Section considers three different sample portfolios, and analyses the reliability of slope-based predictions of the position values. For example, if you are a risk manager aiming to provide Value-at-Risk (VaR) based reporting, then it is very very much less expensive computationally to use the so-called CVaR approach, which predicts portfolio values based on slopes.

It will be seen that in some cases, slope based approximations can fail badly. Though it is also shown that in some such cases alternate methods may be available to provide for sensible trading and risk management.

This type of prediction, in many cases, is an approximation that relates very closely to Taylor Series analysis (see Chapter 8.1.3).

Prediction of position value with slopes for:

- A linear portfolio
- A non-linear monotonic portfolio
- A non-monotonic portfolio

3.4.2.1 A linear portfolio

Figure 3.4 – 3 illustrates the pay-out profile or valuation curve for a linear portfolio. In this case, the slope at underlying market condition A will predict the value at market condition B exactly. Obviously, a linear profile and 1st derivative (i.e. a straight line) are exact match from a "profile" perspective.

[26] For example, those that have "sufficient" continuity in the derivatives (e.g. are sufficiently "smooth"), see Section 3.10.1.

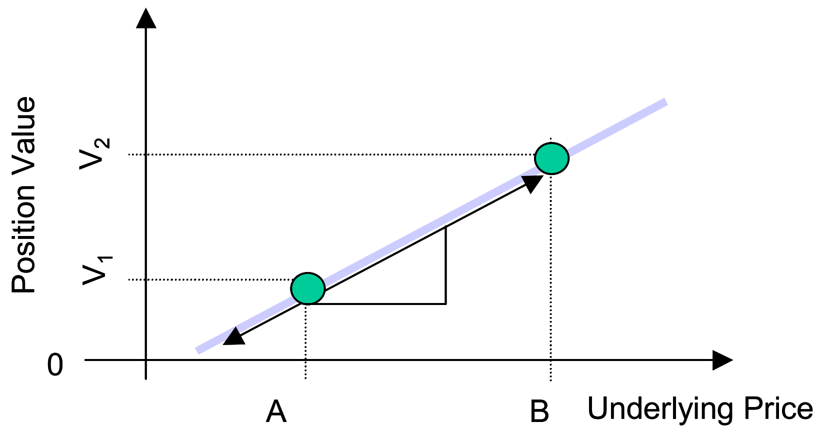

Figure 3.4 – 3. A first order slope is a perfect predictor on a straight line

3.4.2.2 A non-linear monotonic portfolio

In the case of a monotonic non-linear position value, as in Figure 3.4 – 4, the 1st derivative will, in general, only provide an approximation to the true value as the market moves from A to B. However, notice two important results (true for all "well behaved curves"):

- The error might be quite small if there was low curvature (e.g. most bonds have a "sufficiently flat curvature"), or if the market movement was sufficiently small.
- The 2nd (or possibly higher order) derivative(s) at A can be used to modify the prediction, and obtain and exact[27] or "sufficiently" close approximation. This is precisely the notion behind Taylor Series expansion (see Chapter 8.1.3).

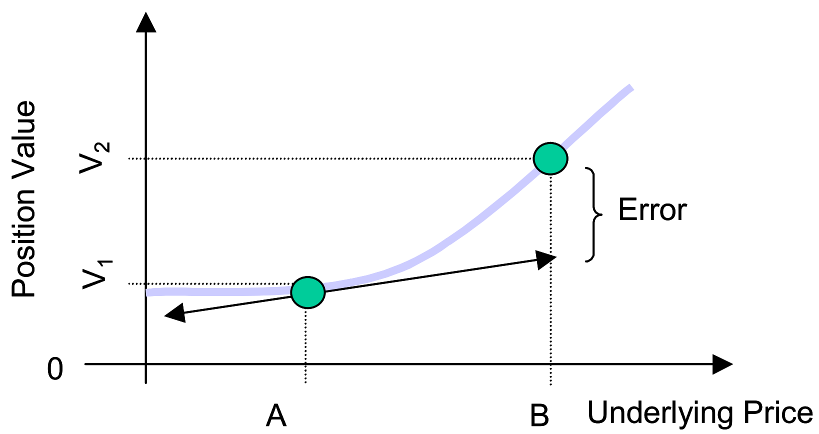

Figure 3.4 – 4. Prediction error when using only a 1st derivative on a "monotonic curved curve"

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[27] If the curvature is exactly 2nd order (such as a parabola), then a combination of a 1st and 2nd derivative based prediction will be exact.