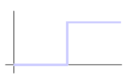
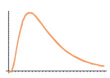
4.8.4 Example Quadrature – Digital Option

Lets revisit the "sample valuation" of a digital option from Section 4.5.5. There it was shown that on expiration the "expected" value of option is the integral of product of the functions composing the distribution and the contracted pay-out formula:

$$E\left[Digital\,Call\right] = \int_{-\infty}^{\infty} \underbrace{\delta\left(X\right)}_{Option\,Pay-Out} \underbrace{\rho\left(x\right)}_{Prob\,Wgt} dx \tag{4.76}$$

and if the Log-Normal distribution is used for $\rho(x)$, then

$$\begin{aligned} E\left[Digital\,Call\right] &= \int_{-\infty}^{\infty} \underbrace{\delta\left(X\right)}_{Option\,Pay-Out} \underbrace{LogN\left(\mu,\sigma\sqrt{t}\right)}_{Prob\,Wgt} dx \\ &= \int_{-\infty}^{\infty} \delta\left(X\right) \frac{1}{x\sigma\sqrt{t2\pi}} e^{-0.5\left(\left(Log[x]-\mu_x\right)/\sigma\sqrt{t}\right)^2} dx \end{aligned} \tag{4.77}$$

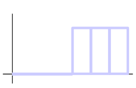
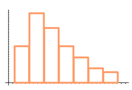
which pictorially is : = $\int$  *  dx

where $\delta\left(X\right)$ is a unit step-function that is zero at all values less than X, and one everywhere else, and so imposes the digital option's payout formula. Thus, here the averaging (via the probability weighting) of the payouts is performed in a single (albeit more complex) calculation step[79].

It was also shown that the integral has an approximation in terms a summation of the product of the probability and pay-out functions:

$$E\left[Digital\,Call\right] \cong \sum_{i=-\infty}^{\infty} \delta\left(X\right) \frac{1}{x\sigma\sqrt{t2\pi}} e^{-0.5\left(\left(Log[x]-\mu_x\right)/\sigma\sqrt{t}\right)^2} \Delta x \tag{4.78}$$

Pictorially, the simplest version of this might look something like:

$\sum$  *  Δx

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[79] So this calculation multiplies together two functions and then integrates (sums) the resultant "product function".

where the "columns" or "bars" have widths of Δx and their heights are given by the respective equations from Equation (11.36) taken at the location (i.e. at the "x" implied by the number of Δx's up to that point).

In fact, this sum (of the product of these functions) composed of "mini-areas" is just a quadrature representation of the continuous solution in Equation (4.77)

Lets apply a real world'ish setting, and consider a digital call option on the S&P index with market conditions as shown in Table 4.8 -1 below. Since this is a digital option, the payouts will be zero where the prices is less than 1550, so the integration (summation) may start at 1550 (as opposed to $-\infty$).

Table 4.8 -1 illustrates a rough approximation using a spreadsheet and an 11-step quadrature approximation of the digital call option on a single forward date. The column marked "***X***" contains the selected forward price points at which the forward probabilities are calculated. The column marked "***Prob***" contains the actual probabilities as per the Log-Normal formula from above[80], please see [3] for more details[81] on quadrature.

				X	***Prob***	***PayOut***	***Product***
Spot	1,418.19	1	F1	2,300	0.00013032	1	0.195487
r	5.66%	1	F2	3,800	0.00006315	1	0.094719
q	0	1	F3	5,300	0.00003575	1	0.053622
Volatility	17.23%	1	F4	6,800	0.00002229	1	0.033428
Time	1	1	F5	8,300	0.00001484	1	0.022261
Strike	1,550.00	1	F6	9,800	0.00001037	1	0.015558
		1	F7	11,300	0.00000752	1	0.011284
XLower	1,550.00	1	F8	12,800	0.00000562	1	0.008429
DeltaX	1500	1	F9	14,300	0.00000430	1	0.006449
Ave	*7.30*	1	F10	15,800	0.00000336	1	0.005034
StdDev	*1.25*	1	F11	17,300	0.00000266	1	0.003997
						Sum	**0.45027**
						PV Sum	**0.425493**

Table 4.8 -1. Spreadsheet approximation using an 11-rectangle quadrature for the forward and present value of a digital call option

[80] In practice, there is more than one "format" of the Log-Normal distribution (due to the choice of units to use for the input parameters etc). Here the so-called "Type-1" formulation is used, which requires that the input average and standard deviations be taken on a "Log" basis.

[81] For example the first entry in the "***Prob***" column is LogNormal(2,300; μ = 7.30, σ = 1.25) = 0.00020569, where the average is calculated as Log[1418.19 * Exp{(0.0566 – $(0.1723)^2$/2)*1yr}] = 7.30, and the standard deviation is calculated as 0.1723* 1yr*Log[1418.19] = 1.25.