6.12 Properties and Manipulation of Distributions

This Section discusses several important properties of distributions, and basic but common manipulations of distributions. These topics include:

- Manipulations relying on the area preservation property.
- Scaling, shifting, stretching, and related transformations.
- Some terminology relating price vs. returns based manipulation.
- The Central Limit Theorem.
- Multiples rules and their derivation for use in place of quantiles.

Other "deeper" properties of distributions and their relationship to moments and moment generating functions are provided in Sections 6.1.3, 6.5.3, and 6.13

6.12.1 Manipulation: Area Preservation as an Auxiliary Equation

The area under any distribution must be 1. This is a basic requirement for probability theory as introduced in Section 6.2. This requirement, though, can also be used as an auxiliary relationship to simplify or otherwise manipulate calculations involving distributions. A very common application of this rule as an auxiliary equation arises as a "complement" calculation in options and VaR related problems.

Suppose that the distribution has been given as the Normal distribution and that probability of exercise or VaR threshold calculation is required. Then,

$$\text{CDF}_{\text{Normal}}\left[x \middle| \mu_x, \sigma_x\right] = \int_{-\infty}^{x} \frac{1}{\sigma_x\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \tag{6.105}$$

but, the complementary probability for x is:

$$\text{CDF}_{\text{Normal}}\left[\widehat{x} \middle| \mu_x, \sigma_x\right] = \int_{x}^{\infty} \frac{1}{\sigma_x\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \tag{6.106}$$

or, relying on the auxiliary relationship that the area under the curve is 1:

$$\text{CDF}_{\text{Normal}}\left[\widehat{x} \middle| \mu_x, \sigma_x\right] = 1 - \int_{-\infty}^{x} \frac{1}{\sigma_x\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \tag{6.107}$$

This result permits the use of the "usual" CDF, and so obviates the need to create a new format for the integration of the "upper quantile".

For example, the value of a digital call option based on a Normally distributed uncertainty and that has a pay-out *V(x)* of 1 when the market is at *x* or above, and 0 otherwise would be:

$$
\begin{aligned}
\mathrm{CDF}_{\mathrm{Normal}}\left[V(x)\middle|\mu_x,\sigma_x\right] &= \int_{-\infty}^{x} V(x)\frac{1}{\sigma_x\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \\
&= \int_{-\infty}^{x} 0\frac{1}{\sigma_x\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx + \int_{x}^{\infty} 1\frac{1}{\sigma_x\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \\
&= \int_{x}^{\infty} \frac{1}{\sigma_x\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx \\
&= 1-\int_{-\infty}^{x} \frac{1}{\sigma_x\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dx
\end{aligned}
\qquad (6.108)
$$

Therefore, the integration of the "upper quantile" has been expressed as a complement of the integration of the lower quantile (i.e. 1 – the integral of the lower quantile), only now the integration is a compound process involving the probability weighted pay-out function (though in this case the weighted pay-out function is quite simple to integrate)

Figure 6.12 – 1 a) illustrate the pay-out function superimposed on a distribution, while b) show the resulting two complementary quantiles.

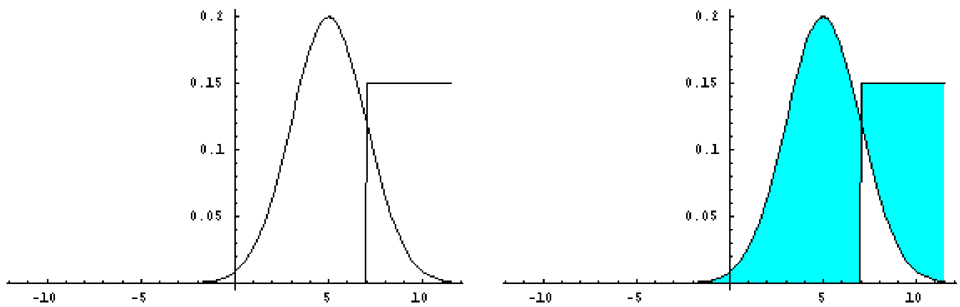


Figure 6.12 – 1. a) The digital option pay-out function superimposed on the Normal PDF , and b) using the complement rule to evaluate the upper quantile (the white area under the curve)

The white region in the right image represents the complementary quantile (i.e. that bit that represent the cumulative probability of a pay-out, or the bit that is "multiplied" by the non-zero portion of the pay-out).

Note that a similar result is available for discrete distributions as well, as shown in Figure 6.12 – 2. This image taken from the earlier discrete digital option pricing example and adapted to the current setting to show a manipulation involving the complementary area under the curve above or to the right of the strike price..

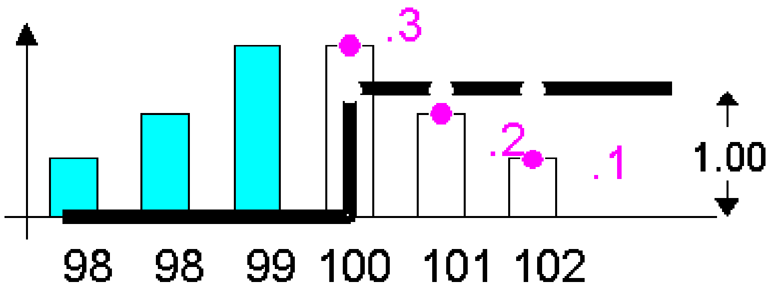
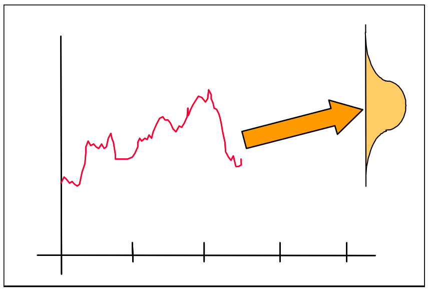


Figure 6.12 – 2. Evaluation of a digital option with pay-out overlaid on a discrete distribution.

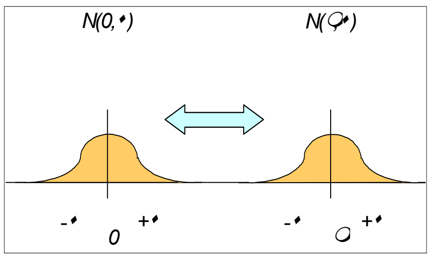
6.12.2 Manipulation: Shifting or De-coupling the Average

Suppose for a moment that the Normal distribution is the "best" shape of uncertainty. The expression for the Normal distribution is $N(\mu,\sigma)$. This has two components: the average and the standard deviation. An expression such as $N(\mu,\sigma)$ is capable of modelling an entire problem on its own, but model development convention often separates the average and variability. This does not cause any difficulties since $N(\mu,\sigma)$ can always be decomposed as. One decomposition is to separate out the average as:



$$N[\mu,\sigma] = \mu + N[0,\sigma] \qquad (6.109)$$

Notice that this decomposition is simply additive, since it is equivalent to "shifting" the centre of the distribution from 0 to μ, as in the figure to the right.



This type of coupling and decoupling of the average and variability is straightforward for many choices of distribution, though in some case the de-coupling from the average may be less obvious (e.g. the Uniform distribution has its end points as input parameters and so manipulation via its moments is an indirect procedure).

This feature of some distributions that permits direct "manipulation by moments" is a very convenient feature for practical modelling and quantitative implementation, as will be seen in Chapters 11 - 13. The Normal distribution is one of the few distributions with this property.