6.13 Moment Generating Functions and the Characteristic Equation

Moment generating functions were briefly discussed in Chapter 5.7 in the context of general moment formulations for dataset analysis. Here, moment generating functions are reviewed briefly in the context of model distributions[310], and to make an important connection to the so-called "characteristic equation[311]".

As the "radiation hazard" symbol indicates, this Section is primarily included for completeness, and most traders and risk managers need only be aware of these matters, as opposed to being able to derive or solve such problems.

General moment generating functions and the characteristic equation do for the variances of distributions something similar to what Taylor Series expansion does for functions. This does not replace Taylor Series expansion. Rather, this is a metaphoric perspective and an additional tool.

The results here permit general derivation of moments for almost any distribution, but see Sections 6.4.3 and 6.5.3 for derived moments of 9 important distributions.

Probabilities, distributions, and moments are related to one another by the characteristic equation of the distribution. The characteristic equation is Fourier transform (a special weighted integral) of the Probability Density Function (PDF). That is, the characteristic equation is a special version of the CDF. If a general expectation for a process X can be defined by the characteristic equation $\phi()$ as:

$$\phi(s) = E\left[is \bullet X\right] = \int_{\Omega} e^{is \bullet x} p(x)dx \tag{6.122}$$

where $X(x) = x$, and s, and X may be vector valued, then the n^{th} moments is generated by:

$$E\left[\prod_i X_i^{m_i}\right] = \prod_i \left(-i\frac{\partial}{\partial s_i}\right)^{m_i} \phi(s)\Big|_{s=0} \tag{6.123}$$

310 Here, the moment generating functions are considered primarily from the continuous distribution perspective, but they are easily amended for the discrete case, as was shown in Chapter 5.

311 In mathematics, the expression "characteristic equation" will arise in many, and to some extent unrelated, contexts.

Equation (6.123) is the generalised moment generation function, since it can be used to derive any one of the moments.

Thus, if the generalised expectation is defined as:

$$\phi_x(t) = E\left[e^{ixt}\right] = \int_{-\infty}^{\infty} e^{ixt} f(x)\,dx \tag{6.124}$$

where *f(x)* is the probability function (and the advanced reader will recognise this to be Fourier transform of the probability density). Then the k^{th} moment is defined as:

$$E\left[x^k\right] = \frac{\phi_x(0)\big|_k}{i^k} \tag{6.125}$$

or for the mean-adjusted version,

$$E\left[(x-\mu)^k\right] = \frac{\phi_{x-\mu}(0)\big|_k}{i^k} \tag{6.126}$$

For example, the simple cases for the mean and variance are:

$$E[X] = \int_a^b xf(x)dx \tag{6.127}$$

and

$$Var[X] = E\left[(x-\mu)^2\right] = \int_a^b (x-\mu)^2 f(x)dx \tag{6.128}$$