6.2.3 Probability Spaces & Algebras (Borel, Sigma, etc)

The modern trend in the teaching of probability theory is to use a set theoretic approach. In particular, most modern treatments of probability theory and stochastic calculus assume that the (mathematical) events take place in a *Borel Space*[238], with a *Sigma-Algebra*.

In mathematics, one can define "spaces" where certain rules hold, or form restrictions. Algebras are rules of operation, often associated with spaces. Does that sound too abstract?

If you were building a house, the rules/restrictions on construction would be different if you were building on solid ground, on an iceberg, or in outer space. Additional, the tools available for construction would also determine what you may or may not build, and how you must go about building.

The technicalities get worse before they get better, since spaces and algebras must be defined in a very general and yet extremely precise sense (after all, precision is what gives mathematics its power).

So lets jump in the deep end, write down the definitions, and then see what in blazes it all means.

Definition – Sigma-Algebra: Given a set X, then a σ-algebra, $\mathbb{F}$, is a non-empty collection of subsets in that set where the following holds:

- $\mathbb{F}$ includes the empty set as an element
- Complements of the set's elements are also in the set
- Given any sequence of elements in $\mathbb{F}$, the union of that sequence is also in $\mathbb{F}$ (closed and bounded)

Lets examine these points: A set X is just a collection of objects or events (e.g. a collection of probabilities for expected prices in a forecast). The σ-algebra $\mathbb{F}$ is just some rules that ensure "sensible" results (albeit at an abstract level). The empty set is included in the set for the same reason that zero is included in the set of real numbers.

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[238] Virtually all mathematics begins with some sort of space. Here a Borel Space is of interest, in other fields one may require a Banach Space, which ensures that functions are twice differentiable/intergrable amongst other things, so that second derivatives (accelerations and diffusions) may exist. In yet other situations, many other types of spaces may be required.

Additionally, if there is an element p_i (e.g. a probability for a forecasted price), then the requirement that its complement is also in the set means that there are elements in the set that account for a probability of $1 - p_i$ (i.e. the complement, e.g. the probability of "not that event").

Finally, for a set (e.g. of probabilities) to be "closed and bounded" means that it is not permitted to use the elements of the set to create a result that falls outside of the set X. If this is a set of probabilities, then all elements are restricted to have values bounded between 0 and 1. Then, $\mathbb{F}$ enforces that manipulation of those elements may not create a probability of, say, 1.4 or –6 (e.g. you are not permitted to make up any old manipulation of the elements to create nonsensical results).

Though this is still at an abstract level, it should be evident now that the $\mathbb{F}$ is created/required when working with probabilities to ensure that the probability machinery is sensible and consistent.

A Borel space is just a set that has many such σ-algebras as subsets. This one is a bit trickier to put into pedestrian terms. However, here is a bit of a "stretch": if you needed to price a derivative that had pay-outs on several forward dates (e.g. a Bermudian option[239]), then you would need probabilities for several different forward dates. Each of those sets of probabilities would have to be in its own subspace, but the ultimate derivative valuation result requires dealing with collection of those subspaces, and hence the need for a Borel space.

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[239] A Bermudian option has pay-out/exercise possibilities according to s fixed schedule of dates, as compared with a European option that may be exercised on only one date, and as compared to an American option that may be exercised at any time (an "infinite number of dates", sort of). See [8] for detailed discussions on exercise styles for options.