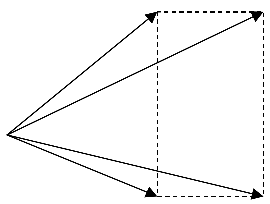
6.6 Multi-Dimensional Distributions

Multi-dimensional, also known as multi-variate, processes require multi-dimensional distributions. A vanilla convertible bond (CB) is influenced by both interest rates, and equity prices, and so is driven by a bivariate uncertainty. If that CB is denominated in a foreign currency, then it may be considered a trivariate problem[259].

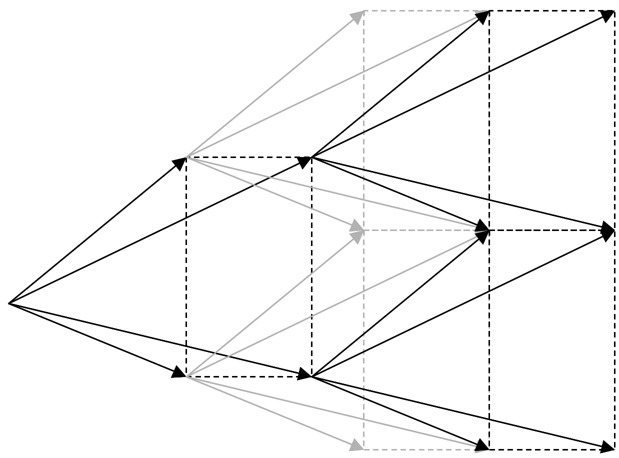
Multi-dimension distributions are defined for virtually any of the important “shapes”. However, the practical implementation and usage can be disproportionately greater than the univariate case

6.6.1 Example: Discrete Distributions in 2- Dimensions

The terminology may be a bit confusing here since a distribution (such as the Binomial distribution) can be univariate, bivariate, multi-variate etc. The “nomial” bit refers to the number of outcomes in a single draw, while the “variate” bit refers to the number of dimensions. The image below shows a schematic for a one-step bivariate Binomial process (which has four outcomes since there are two for each dimension). The image to the right shows a schematic for two steps with a bivariate Binomial process. Think of these as “horizontal pyramids”. Clearly, the number of outcomes (9 for the second step) will grow extremely quickly with number of trials or steps[260].



One-step Bivariate Binomial



Two-step Bivariate Binomial

[259] This depends how the FX component is structured, and may require IRP considerations. Also, there are CBs that are called CBs but are effectively a forward on the equity, in which case there is no “optionality” and so it may be only a univariate problem from an option perspective.

[260] Indeed, the number of outcomes can grow very much more quickly than shown here, since it is also possible to have non-recombining bivariate Binomial, which would have 16 outcomes at that second trial/step. This matter is discussed in more detail in the Trees/Lattice methods Chapter 16.

In trading, the "variates" are most often associated with underlying prices or indices. By analogy to the CBs discussion above, the points in the 2-D Binomial process shown above may be considered to be, say, the interest rate along the vertical axis, equity prices along the axis going "into" the page, and time along the horizontal axis.

Notice that the probabilities must now define "*p*" and "*q*" for each variate (dimension or price or factor) and this will also introduce correlations or cross-terms. The details of the bivariate Binomial are bit tedious and beyond the scope of this Chapter, and so are deferred to [3.c], but one possible approach[261] to calculating the probabilities in the bivariate Binomial case is:

$$\text{Prob}_{Bi-B}\left[X,Y\right]=\begin{cases} p_{UpDown}=\dfrac{1}{2}\left(\dfrac{n!}{j!(n-j)!}\right) \\ p_{InOut}=\dfrac{1}{2}\left(\dfrac{n!}{k!(n-k)!}\right) \end{cases} \tag{6.86}$$

where the index *j* denotes nodes in the UpDown dimension, and the index *k* denotes nodes as arrived at in a horizontal InOut direction, see [3.c] for examples of implementation.

A bivariate Poisson probability function has the form:

$$\text{Prob}_{Bi-P}\left[X,Y\right]=e^{-(\theta_1+\theta_2+\theta_3)}\frac{\theta_1^x}{x!}\frac{\theta_2^y}{y!}\sum_{i=0}^{\min(x,y)}\binom{x}{i}\binom{y}{i}i!\left(\frac{\theta_3}{\theta_1\theta_2}\right)^i \tag{6.87}$$

In some cases, multi-dimensional process may take on one type of distribution along one dimension and another type of distribution along another dimension, such as a bivariate Binomial-Poisson distribution. The derivation and construction of these mixed cases can be very involved.

[261] This approach relies on reconstructing the bivariate process so that in can be generated from two univariate components, as derived in [24]. There, the results were applied to the valuation of Rainbow Options (see [9] for details of exotic options).

6.6.2 Example: Continuous Distributions in 2- Dimensions

The Bivariate Normal distribution is an example of a 2-dimensional continuous distribution. The probability density function can be written as:

$$\text{Prob}_{Bi-N}\left[X,Y\right]=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}e^{\left[\frac{-1}{2\left(1-\rho_{X,Y}^2\right)}\left(\frac{\left(X-\mu_X\right)^2}{\sigma_X^2}-2\frac{\rho_{X,Y}\left(X-\mu_X\right)\left(Y-\mu_Y\right)}{\sigma_X\sigma_Y}+\frac{\left(Y-\mu_Y\right)^2}{\sigma_Y^2}\right)\right]} \quad (6.88)$$

The second term in the exponential should be recognisable as a kind of portfolio variance for a portfolio that is composed of two components: *X* and *Y* (e.g. two bonds, or a bond and an equity, or whatever). This expression could also be written in matrix notation, as illustrated in Section 5.6.6. That is, this second term is the result of a covariance matrix multiplication.

Just as with the univariate version, this cumulative probability is expressed in terms of (multi-dimensional) *Erf[X,Y]*, which is not intergrable analytically, and so special approximations are used (see [3.a]). The charts in Section 5.6 were created with that type of approximation to the integral of Equation (6.88), which leads to results as shown in Figure 6.6 – 1.

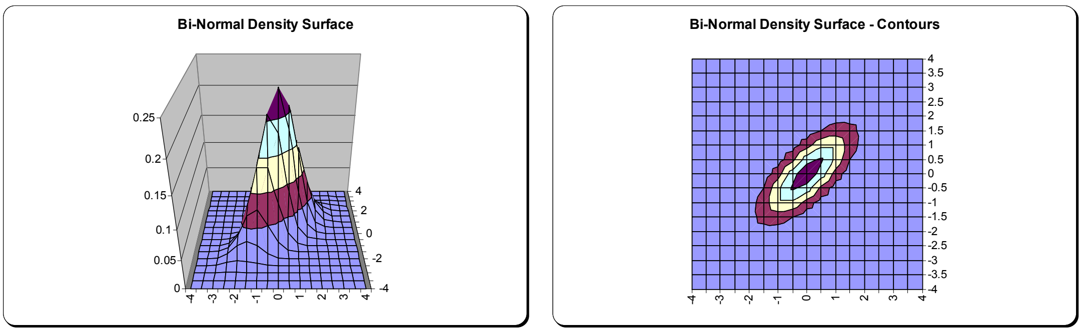


Figure 6.6 – 1. Bivariate Normal distribution with correlation coefficient 0.75