8 Expanding the Complex

This Chapter provide a very short review of (possibly infinite) series representation and expansion of functions. Full coverage of this topic is too technical and abstract, and for trading floor purposes, only a working knowledge of the basic representations is essential.

Series representation and expansion embodies one of the most important principals in mathematics: transformation of complex problems into a collection of simpler problems, or conversely, building complex solutions with a many simple pieces. Only a few noteworthy topics will be reviewed, including:

- Series expansion and Series representation
- Polynomial and Power Series Representation
- Trigonometric Series Representation
- Taylor Series (Derivatives Series Representation)
- Digression on aperiodic problems
- Big O, Little O, and Analyticity

8.1 Series Expansion & Representation

Much of the following has the look and feel of a "curve fitting", and that is a useful metaphor. However, series representation and expansion intends to create the "exact" function, and even when only a partial series is used to approximate a function, it is built on the foundation of exact representation, rather than "something that looks close enough".

Two important (approximation) methods for converting complex problems into an equivalent or approximate collection of simple building blocks are piece-wise simplifying methods (this includes most the classic numerical methods such as quadrature, and Finite Difference), and series representations methods. Figure 8.1 – 1 illustrates that the piece wise methods represent a complex problem (e.g. a curve) with a sequence of simple building blocks (e.g. straight-lines) referred to as basis curves, which are each joined in a piece-wise sense[333].

[333] The mathematics text may sometimes use an expression such as "a function having a compact support, and non-empty on a small set". That is just "fancy speak" for a "bunch of little pieces connected to make a big piece".

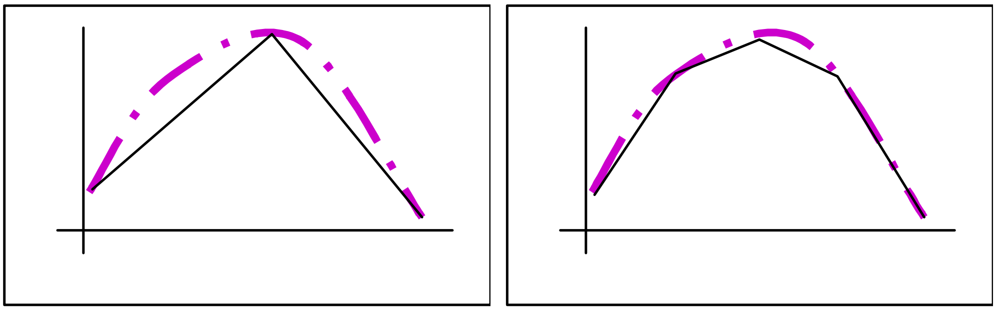


Figure 8.1 – 1 a) Piece-wise approximation of the “exact answer” with two straight-lint segments, and b) an approximation with many straight-line segments.

Series representation takes a different approach, as illustrated in Figure 8.1 – 2. Instead of using piece-wise building blocks to cover neighbouring segments, rather use building blocks that each cover the entire “range” of the problem, but are built using “different amounts” of simple “curves”. Figure 8.1 – 2 shows some weighted combination of simple polynomials, referred to as basis curves, that span the space of interest, can reproduce quite general and complex curves.

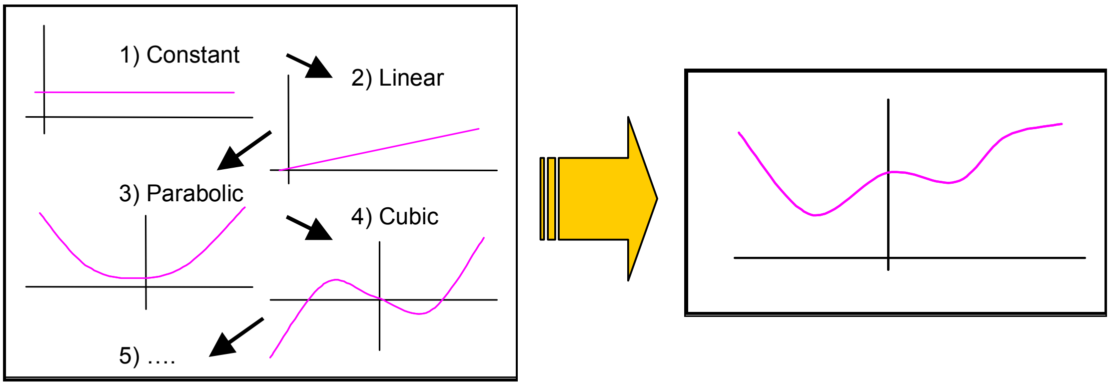


Figure 8.1 –.2 Illustration of creating a complex curve of (virtually) any shape by adding together “different amounts” of a sequence of (simpler) basis curves.

This is later method is a series representation when the component functions form a mathematical series, such as a sequence of polynomials. Sometimes, different types of mathematical functions can be mixed together to obtain special features, but the basic series representation tends to use families of basis functions.

In different words, the piece-wise methods join neighbouring basis curves to form a collection representing the target function, while series representation superimposes (weighted) basis functions to produce the target curve.

Each family of basis functions has specific properties that lend themselves better to specific class of problems, though often it is possible to construct the target curve by any of the families of basis functions. The key families of basis functions for series representation considered here are:

- Power Series Polynomials
- Trigonometric Functions
- Derivatives

There are technical considerations and restrictions for the applicability of not only the choice of the family of basis function, but also whether series expansion is tractable. That is:

- There are methods for assessing the representation approximation error, and also methods for estimating the "convergence" of the series representation to the "true" functional form, such as Big O and Little O methods.
- Also, there are functions that cannot be replicated in any sensible or reliable way with series representation, such non-analytical or aperiodic/singular functions.

On the other hand, sometimes series representation is exact. Indeed, in some cases the series representation can be more efficient or convenient than the fundamental solution.

Thus, in general, series representation has the form

$$f(x) = \sum_i w_i \theta_i \tag{8.1}$$

where the w_i are weighting coefficient or functions, and the θi are the basis functions. The representation may be multi-dimensional, and may require infinite series.

Caveat: while it is helpful to think of series expansion as a kind of curve fitting process, in the present context it is something considerable more. That is, while curve fitting says "I can create something that is close enough", series representation says "you have to create the exact function, even if you end up only using part of it".

8.1.1 Polynomial & Power Series Representation

Polynomial, and especially power series polynomial, series form a very convenient and powerful family of basis functions. A 1-D power series polynomial has the form:

$$f(x) = \sum_i w_i \theta_i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$
$$= \sum_{i=0}^{\infty} a_i x^i \tag{8.2}$$

In 2-dimensions, this takes the form:

$$\begin{aligned} f(x,y) = \sum_{i,j} w_{i,j} \theta_{i,j} &= a_{0,0} + a_{1,0} x + a_{2,0} x^2 + a_{3,0} x^3 + \ldots \\ &+ a_{1,1} x\ y + a_{2,1} x^2 y + a_{3,1} x^3 y + \ldots \\ &+ a_{1,2} x\ y^2 + a_{1,3} x\ y^3 + a_{1,4} x\ y^4 + \ldots \\ &+ \ldots. \\ = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{i,j} x^i y^j \end{aligned} \tag{8.3}$$

These formulations are composed of increasing order of basis curves such as a constant (a_0), a straight-line ($a_1 x$), a parabola ($a_2 x^2$), and so forth, as shown in Figure 8.1 – 3 a) – d). The coefficients (the a_i's) are the weights or amounts of each basis curve.

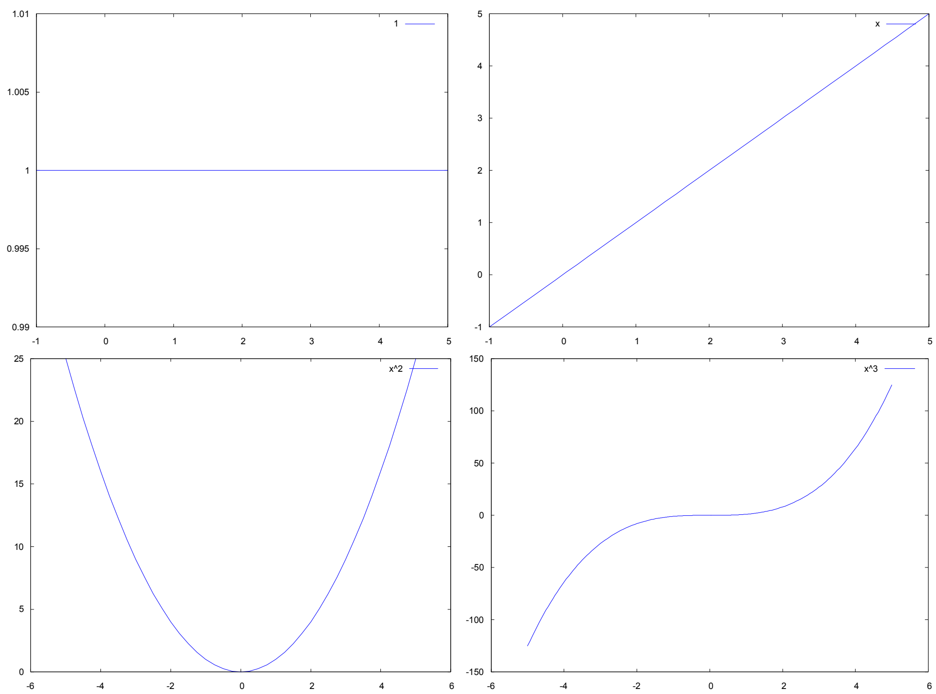
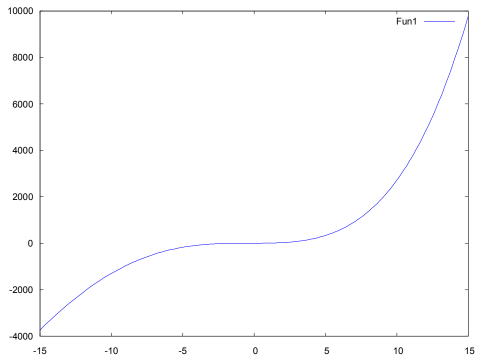
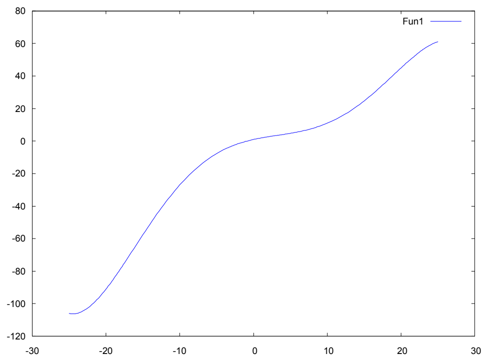


Figure 8.1 – 3 A sequence of power series polynomial basis curves corresponding to a_0, $a_1 x$, a_2, x^2, and a_3, x^3 in Equation (8.2)

Equation (8.2) can be chosen with whatever component basis curves are appropriate, and with appropriate values for the coefficients to produce a wide range of very general curves as illustrated in Figure 8.1 – 4, though see Section 8.2.



f(x) = 1+x+2*x^2+2*x^3+.051*x^4



f(x) = 1+x-.1*x^2+.01*x^3+.0001*x^4-.00001*x^5

Figure 8.1 – 4. Illustration of creating a complex curve of (virtually) any shape by adding together “different amounts” of a sequence of (simpler) “basis” curves.